

1975

Infinite-Dimensional Topology and the Theory of Shape.

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CERIN, Zvonko Tomislav, 1950-
INFINITE-DIMENSIONAL TOPOLOGY AND THE
THEORY OF SHAPE.

The Louisiana State University and Agricultural
and Mechanical College, Ph.D., 1975
Mathematics

Xerox University Microfilms, Ann Arbor, Michigan 48106

INFINITE-DIMENSIONAL TOPOLOGY AND THE THEORY OF SHAPE

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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December, 1975

ACKNOWLEDGMENT

The author wishes to express his appreciation to Professor R. D. Anderson for his advice, criticism and encouragement during the past three years and especially during the preparation of this dissertation. He is also grateful to Professors R. M. Schori and D. W. Curtis at L.S.U. for their friendly care and encouragement, and J. E. West at Cornell and T. A. Chapman at the University of Kentucky for their hospitality during his stay there.

I also wish to acknowledge my gratitude to Professor G. Kozlowski who read the manuscript and made several improvements of the text.

Finally, I wish to thank Monica Loftin for her excellent typing job.

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ABSTRACT

The tools of infinite-dimensional topology were frequently used in the development of shape theory. The most remarkable examples of this are Chapman's complement theorems which show: (a) that the study of shape properties of compacta is equivalent to the study of weak proper homotopy of open subsets V of the Hilbert cube Q for which $Q-V$ is a Z -set in Q , and (b) that two Z -sets $X, Y \subset Q$ have the same shape if and only if their complements $Q-X$ and $Q-Y$ are homeomorphic.

We shall apply results (a) and (b) in the investigation of various shape invariants of compacta. The natural extensions of this simple idea lead us to the study of three different but closely interrelated topics described below.

¹ We shall rework a large portion of shape theory of compacta using techniques of infinite-dimensional topology. This is done as follows. Let α be a shape invariant property of compacta (say, movability). Then we can define a property $\tilde{\alpha}$ of all non-compact locally compact metric spaces such that the statement "A Z -set X in Q has property α if and

only if the complement $Q-X$ of X has property \tilde{a} holds. In many instances it is relatively easier to investigate property \tilde{a} which makes most of our proofs simpler and more geometric than the original ones. In Chapters III, IV, V, VI and VII this program will be carried on for shape dimension, movability, shape absolute neighborhood retracts, shape absolute retracts, and UVW^n -spaces, respectively.

2° We shall introduce the category \mathcal{LC}_∞ of non-compact locally compact metric spaces and classes of homotopic at ∞ proper maps between them and study the homotopy theory of that category. In the short Chapter VIII, we will show how objects of \mathcal{LC}_∞ can be considered as inverse systems and how classes of homotopic at ∞ proper maps between one-ended locally compact ANR's induce homotopy classes of maps of associated systems. Chapter IX treats the question: "When is a proper map $f:M \rightarrow N$ between non-compact locally compact metric spaces a homotopy equivalence at ∞ ?" Our conditions are in terms of two geometric properties of f detectable by maps of polyhedra. In Chapters III, IV, V, and VI, property \tilde{a} , discussed above, is considered as a part of homotopy theory of \mathcal{LC}_∞ .

3° Relationships between 1° and 2° are exploited throughout.

Chapter II contains the proof of the relative version of the Geometric Characterization Theorem of Chapman and

Siebenmann.

In Chapter I, we collect relevant definitions and results from infinite-dimensional topology in order to make our presentation self-contained as much as possible.

CHAPTER I

INTRODUCTION AND PRELIMINARIES

The notion of shape, introduced by Borsuk [5], is a modification of homotopy type designed to allow the study of global properties of locally complicated compacta for which the classical theory does not give much information. Later on, several people including Mardesić-Segal [36], Fox [25], Kozłowski [29], Mardesić [33], and Ball-Sher [4] extended Borsuk's theory of shape to various classes of spaces. In this dissertation, we shall consider shapes of compact metric spaces and of locally compact metric spaces. It is well known [33] that in this category of compacta, in the absolute case, all standard approaches coincide.

The tools of infinite-dimensional topology were frequently used in the development of shape theory. The most remarkable examples of this are Theorems 1 and 2 in [12], where Chapman showed (see later on in this chapter for precise statements): (a) that the study of shape properties of compacta is equivalent to the study of weak proper homo-

topology of open subsets V of the Hilbert cube Q for which $Q-V$ is a Z -set in Q (for definitions, see later on in this chapter), and (b) that two Z -sets $X, Y \subset Q$ have the same shape if and only if their complements $Q-X$ and $Q-Y$ are homeomorphic.

It was suggested by R. D. Anderson that by applying results (a) and (b) one can get new ways of treating shape invariants of compacta like shape dimension, movability, shape absolute (neighborhood) retracts, and others. The nucleus of the main idea in this dissertation is that suggestion. The natural extensions of it lead us to the study of four different but closely interrelated topics described below.

1⁰ We shall rework a large portion of shape theory of compacta using techniques of infinite-dimensional topology. This is done as follows. Let α be a shape invariant property of compacta (say, movability). Then we can define a property $\tilde{\alpha}$ of all non-compact locally compact metric spaces such that the statement "A Z -set X in Q has property α if and only if the complement $Q-X$ of X has property $\tilde{\alpha}$ " holds. In many instances it is relatively easier to investigate property $\tilde{\alpha}$ which makes most of our proofs simpler and more geometric than the original ones. In Chapters III, IV, V, VI, and VII, this program will be carried on for shape dimension, movability, shape absolute neighborhood retracts,

shape absolute retracts, and UVW^n -spaces, respectively.

2^o We shall introduce the category \mathcal{LC}_∞ of non-compact locally compact metric spaces and classes of homotopic-at- ∞ proper maps between them and study the homotopy theory of that category. The concept of homotopy-at- ∞ is weaker than Chapman's weak proper homotopy (see Chapter I) and it seems more useful. In the short Chapter VIII, we will show how objects of \mathcal{LC}_∞ can be considered as inverse systems and how classes of homotopic-at- ∞ proper maps between one-ended locally compact ANR's induce homotopy classes of maps of associated systems. Chapter XI treats the question: "When is a proper map $f:M \rightarrow N$ between non-compact locally compact metric spaces a homotopy equivalence at ∞ ?" Our conditions are in terms of two geometric properties of f detectable by maps of polyhedra. In Chapters III, IV, V, and VI, property $\tilde{\alpha}$, discussed above, is considered as a part of homotopy theory of \mathcal{LC}_∞ .

3^o Relationships between 1^o and 2^o are exploited throughout.

4^o Let us observe that another aspect of our work can be interpreted as a study of proper shape theory of non-compact locally compact metric spaces, due to Ball and Sher [4], by considering Z-sets in $Q - \{\text{point}\}$ instead of in Q .

This dissertation is only an initial step in the direction of effective use of ideas 1^o-4^o and the author

plans to continue research on these questions.

Chapter II contains the proof of the relative version of the Geometric Characterization Theorem of Chapman and Siebenmann [17]. We include the proof for two reasons, first, this theorem has a key role in our program described under 1^o so that the understanding of its proof is rather important in later chapters, and second, since it is proved in the relative form, most of our discussion in the remaining chapters holds for pairs of compact metric spaces as well. However, we warn the reader that in this case our results concern a shape of pairs of compact metric spaces in terms of relative ANR-sequences defined by Mardesić and Segal [36].

In this chapter, we collect relevant definitions and results from infinite-dimensional topology in order to make our presentation self-contained as much as possible.

In this dissertation all statements are numbered with two numbers so that the first number indicates a chapter in which it is contained. For example, Theorem 1.3 means the third numbered statement in Chapter I, and Corollary 3.5 is the fifth result in Chapter III.

The most often used definitions and results from infinite dimensional topology will be listed here. We shall assume that the reader is familiar with the shape theory of compacta in the description using ANR-sequences given by Mardesić and Segal [36]. It is known [37] that in the

absolute case this approach is equivalent to Borsuk's original definition [5], but in the relative case they are different [35].

The Hilbert cube Q is represented as $\prod_{i>0} I_i$, where $I_i = [-1,1]$. For each $n > 0$ we can write $Q = I^n \times Q_{n+1}$ with $I^n = I_1 \times \cdots \times I_n$ and $Q_{n+1} = I_{n+1} \times I_{n+2} \times \cdots$. The endslice $\{1\} \times Q_2$ is denoted by W . By a prism neighborhood of a subset $A \subset Q$ we mean any neighborhood of the form $P \times Q_{n+1}$ where $P \subset I^n$ is a polyhedron. A Q-manifold is a separable metric manifold modeled on Q . A closed subset X in a Q -manifold M is called a Z-set in M provided for every non-empty homotopically trivial open subset U of M , $U-X$ is also non-empty and homotopically trivial. The notion of Z-set was introduced by Anderson [1] and plays an important role in infinite-dimensional topology primarily because of the following mapping replacement and Z-set unknotting theorems of Anderson and Chapman [2] (a useful relative version of the latter is stated as Lemma 2.3 below and is due to Felt [23]).

Theorem 1.1 (Mapping Replacement). Let M be a Q -manifold, \mathcal{U} an open cover of M , X be a locally compact separable metric space, and let Y be a closed subset of X . If $f: X \rightarrow M$ is any proper map such that $f|_Y$ is an embedding of Y onto a Z-set in M , then there is

an embedding $g: X \rightarrow M$ such that $g(X)$ is a Z -set ,
 $g|_Y = f|_Y$, and g is $St(\mathcal{U})$ -close to f .

Theorem 1.2 (Z -set Unknotting). Let M be a Q -manifold, \mathcal{U} an open cover of M , X a locally compact separable metric space and $F: X \times [0,1] \rightarrow M$ a proper homotopy limited by \mathcal{U} such that F_0 and F_1 are embeddings of X onto Z -sets in M . Then there is an ambient isotopy $G: M \times I \rightarrow M$ limited by $St^4(\mathcal{U})$ such that $G_1 \circ F_0 = F_1$ (ambient means that each G_t is a homeomorphism onto).

Recall that a map $f: X \rightarrow Y$ between locally compact metric spaces is proper if $f^{-1}(A)$ is compact for every compact subset A of Y .

Following Chapman [12], two proper maps $f, g: X \rightarrow Y$ are weakly proper homotopic if for every compactum $B \subset Y$ there is a compactum $A \subset X$ and a homotopy $H: X \times I \rightarrow Y$ joining f and g with $H((X-A) \times I) \subset Y-B$. This is an equivalence relation on the class of all proper maps $X \rightarrow Y$ and in a usual way one defines notions of a weak proper homotopy equivalence and a weak proper homotopy domination.

Now, we can state precisely Chapman's Theorems 1 and 2 from [12].

Let \mathcal{S} be the shape category of Z -sets in Q and homotopy classes of fundamental sequences between them, and let \mathcal{O} be the category whose objects are complements of

Z-sets in \mathcal{Q} and whose morphisms are weak proper homotopy classes of proper maps between them.

Theorem 1.3. There is a category isomorphism $T:\mathcal{J} \rightarrow \mathcal{Q}$ such that $T(X) = Q-X$ for every object X of \mathcal{J} .

Theorem 1.4. Z-sets X and Y in \mathcal{Q} have the same shape if and only if $Q-X$ and $Q-Y$ are homeomorphic (in notation, $Q-X \cong Q-Y$).

We shall mostly work with a larger category \mathcal{LC}_∞ of locally compact metric spaces and classes of proper maps between them that are homotopic near ∞ , defined as follows. Proper maps $f, g: X \rightarrow Y$ are homotopic-at- ∞ provided for any compactum $B \subset Y$ there is a compactum $A \subset X$ and a homotopy $H: (X-A) \times I \rightarrow Y-B$ joining $f|_{X-A}$ with $g|_{X-A}$. Again, this is an equivalence relation on the class of all proper maps $X \rightarrow Y$ so we can define notions of a homotopy equivalence-at- ∞ and a homotopy domination-at- ∞ . Let us observe that in case Y is an absolute retract, if proper maps $f, g: X \rightarrow Y$ are homotopic-at- ∞ , then they are weak proper homotopic. Therefore, in Theorem 1.3, we can replace weak proper homotopy classes of proper maps by classes of homotopic-at- ∞ proper maps.

A mapping cylinder $M(f)$ of a map $f: X \rightarrow Y$ between compact metric spaces is obtained from the disjoint union of $X \times [0, 1]$ and Y by identifying $(x, 1)$ with $f(x)$

for every $x \in X$. A natural copy of Y in $M(f)$ is the base of $M(f)$ and $X \times \{0\} \subset M(f)$ is the top of $M(f)$. If $\sigma = \{X_i, f_i\}_{i=1}^{\infty}$ is an inverse sequence of compact metric spaces then the infinite mapping cylinder $\text{Map}(\sigma)$ of σ is a result of piecing $M(f_i)$'s together, i.e., $\text{Map}(\sigma) = (X_1 \times \{1\} \cup \bigcup_{i>1} X_i \times [i-1, i]) / \sim$, where $(x, i) \sim (f_i(x), i)$ for all $x \in X_{i+1}$ and $i > 0$. All points of $\text{Map}(\sigma)$ with the second coordinate $\geq k$ are denoted by $\text{Map}_k(\sigma)$. It is clear how to define $\text{Map}(\sigma, \sigma_0)$, where $\sigma = \{(X_i, X_{0i}), f_i\}_{i=1}^{\infty}$ is an inverse sequence of compact pairs (X_i, X_{0i}) and maps of pairs f_i and $\sigma_0 = \{X_{0i}, f_i|_{X_{0i+1}}\}_{i=1}^{\infty}$.

Infinite mapping cylinders have been used often in topology and their usefulness will become evident in Chapters II and III. There we shall repeatedly need Chapman's Cylinder Completion Theorem [17].

Theorem 1.5 (Cylinder Completion). If $\sigma = \{X_i, f_i\}_{i=1}^{\infty}$ is an inverse sequence of compact polyhedra X_i with inverse limit $\varprojlim \sigma$, then $(\text{Map}(\sigma) \cup \varprojlim \sigma) \times Q$ is a compact Q -manifold homeomorphic to $X_1 \times Q$.

Finally, some comments on maps that we shall work with. By a homeomorphism we shall always mean a homeomorphism onto and a homeomorphism into is called an embedding. A CE-map is a proper surjection all point inverses of which have trivial shape.

CHAPTER II

RELATIVE GEOMETRIC CHARACTERIZATION THEOREM

In this chapter, we shall prove the relative version (Theorem 2.1) of the Geometric Characterization Theorem due to Chapman and Siebenmann [17].

By a Q-manifold pair (M, M_0) we mean that M and M_0 are Q-manifolds with M_0 a Z-set in M . We will say that a pair (Z, Z_0) of compacta is a boundary for a Q-manifold pair (M, M_0) if there exists a compact Q-manifold pair $(N, N_0) \supset (M, M_0)$ such that $(N, N_0) - (M, M_0) \cong (Z, Z_0)$ is a Z-pair in (N, N_0) , i.e., $N - M$ is a Z-set in N and $N_0 - M_0$ is a Z-set in N_0 .

The above definition of a boundary for a Q-manifold pair is a relativization of a corresponding absolute form in [17]. In fact, using results like Lemmas 2.2 - 2.6 and Theorem 2.7 below, it is possible to extend most of the theorems from [17] to our setting.

Theorem 2.1. A Q-manifold pair (M, M_0) admits a boundary if and only if there is a relative inverse

sequence $(\sigma, \sigma_0) = \{X_i, Y_i, f_i\}$ of compact polyhedra X_i and closed subpolyhedra $Y_i \subseteq X_i$ such that $\text{Map}(\sigma, \sigma_0) \times Q \cong (M, M_0)$.

If $(M, M_0) \cong \text{Map}(\sigma, \sigma_0) \times Q$ then the pair $(\lim \sigma, \lim \sigma_0) \times Q$ is a boundary for (M, M_0) by the Cylinder Completion Theorem 1.5.

The other half of Theorem 2.1 is proved by a method given in [17] with special attention to certain Z-sets.

We will first state several standard results from infinite-dimensional topology in the relative setting that are needed for a successful relativization of Chapman-Siebenmann's proof.

Lemma 2.2 (Chapman [13]). Every compact Q-manifold pair (M, M_0) can be triangulated, i.e., if K_0 is a finite simplicial complex and $\varphi: M_0 \rightarrow K_0 \times Q$ is a homeomorphism, then φ can be extended to a homeomorphism ψ of M onto $K \times Q$, where K is a finite simplicial complex containing K_0 as a PL subspace.

Lemma 2.3 (Felt [23]). Let (M, M_0) be a Q-manifold pair and X_0 a closed subspace of a space X . Let \mathcal{U} be an open cover of M and let $F: (X, X_0) \times [0, 1] \rightarrow (M, M_0)$ be a proper homotopy limited by \mathcal{U} such that F_0 and F_1 are embeddings of (X, X_0) onto Z-pairs of (M, M_0) . Then there exists a homeomorphism $h: (M, M_0) \rightarrow (M, M_0)$ limited

by $\text{St}^{400}(\mathcal{U})$ such that $h \circ F_0 = F_1$.

Lemma 2.4. Let $f: (M, M_0) \rightarrow (N, N_0)$ be a CE-mapping between compact Q-manifold pairs such that $f^{-1}(N_0) = M_0$ and let $\epsilon > 0$. Then there is a homeomorphism $h: (M, M_0) \rightarrow (N, N_0)$ that is ϵ -close to f .

Proof. Let $\eta, \frac{\epsilon}{5} > \eta > 0$, have the property that 2η -close maps into N are $\epsilon/5$ -homotopic. Since CE-maps of compact Q-manifolds can be uniformly approximated by homeomorphisms [15], there are homeomorphisms $h': M \rightarrow N$ and $h'': M_0 \rightarrow N_0$ that are η -close to f and $f|_{M_0}$, respectively. By Z-set unknotting, there is a homeomorphism $G: N \rightarrow N$ $4\epsilon/5$ -close to the identity and such that $G \circ h'|_{M_0} = h''$. Put $h = G \circ h'$. \square

Lemma 2.5. Let $f: (M, M_0) \rightarrow (N, N_0)$ be a CE-map between compact Q-manifold pairs such that $f^{-1}(N_0) = M_0$. Then there is a homotopy $H: (M, M_0) \times I \rightarrow (N, N_0)$ such that $H_0 = f$ and H_1 is a homeomorphism $(M, M_0) \rightarrow (N, N_0)$.

Proof. Let \mathcal{U} be the normal cover for the open subset $N \times [0, 1)$ of $N \times [0, 1]$. Let \mathcal{V} be a refinement of \mathcal{U} with the property that $\text{St}^4(\mathcal{V})$ also refines \mathcal{U} . Observe that for every $t \in [0, 1)$ there is $1 > \epsilon_t > 0$ with the property: if $x, y \in N$ are such that $d(x, y) < \epsilon_t$ then (x, t) and (y, t) are in the same member of \mathcal{V} ,

where \mathcal{V} refines \mathcal{V} and any two \mathcal{V} -close maps into $N \times [0,1)$ are \mathcal{V} -homotopic. Clearly, we can define a continuous map $\epsilon: [0,1) \rightarrow (0,1)$ such that $\epsilon(t) < \epsilon_t$ for every $t \in [0,1)$.

Since groups of homeomorphisms $\mathcal{K}(M,N)$ and $\mathcal{K}(M_0,N_0)$ are locally contractible [14], and CE-maps between compact Q-manifolds can be uniformly approximated by homeomorphisms [15], using Lemma 2.4, we can find arcs h_t and g_t such that $h_1 = f$, $g_1 = f|_{M_0}$, $h_t \in \mathcal{K}(M,N)$ and $g_t \in \mathcal{K}(M_0,N_0)$ for all $t \neq 1$, $d(h_t, f) < \epsilon(t)/2$, $d(g_t, f|_{M_0}) < \epsilon(t)/2$, and $g_0 = h_0|_{M_0}$.

Define $H: M \times I \rightarrow N \times I$ by $H(m,t) = (h_t(m), t)$ and $G: M_0 \times I \rightarrow N_0 \times I$ by $G(m_0,t) = (g_t(m_0), t)$. Observe that $H|_{M_0 \times [0,1)}$ and $G|_{M_0 \times [0,1)}$ are two proper embeddings onto Z-sets in $N \times [0,1)$ that are \mathcal{V} -close and therefore \mathcal{V} -homotopic. If \mathcal{V} is small enough this homotopy will be proper. By Z-set unknotting there is a homeomorphism $F: N \times [0,1) \rightarrow N \times [0,1)$ limited by $St^4(\mathcal{V})$ such that $F \circ H|_{M_0 \times [0,1)} = G|_{M_0 \times [0,1)}$. Then

$$\tilde{H} = \pi_N \circ F^* \circ H: M \times I \rightarrow N \times I \rightarrow N \times I \rightarrow N$$

is the required homotopy, where F^* is a natural extension of F . \square

Lemma 2.6. Let $(N, N_0) \subseteq (M, M_0)$ be a compact

Q -manifold pair in a Q -manifold pair (M, M_0) such that N is a Z -set in M and N_0 is a Z -set in M_0 . Then there is a collar $\varphi: N \times [0, 1) \rightarrow M$ of N , i.e., an open embedding with $\varphi|_{N \times 0} = \text{id}$, such that $\varphi(N_0 \times [0, 1)) \subset M_0$.

Proof. Since N is a Z -set Q -manifold in M , there is an open embedding $\varphi': N \times (0, 2) \rightarrow M$ such that $\varphi'(n, 0) = n$ for every $n \in N$. Now we are going to change φ' so as to achieve the required φ . Observe that $N \cup \varphi'(N_0 \times [0, 1))$ is a Z -set in M . Indeed, $\varphi'(N_0 \times (0, 2))$ is a Z -set in $\varphi'(N \times (0, 2))$ because N_0 is a Z -set in N ; hence a closed subset $\varphi'(N_0 \times [1/i, 1))$ of $\varphi'(N_0 \times (0, 2))$ is a Z -set in $\varphi'(N \times (0, 2))$ which is an open subset of M so that $\varphi'(N_0 \times [1/i, 1))$ is a Z -set in M . Then $N \cup \varphi'(N_0 \times [0, 1)) = N \cup \bigcup_{i=1}^{\infty} (\varphi'(N_0 \times [\frac{1}{i}, 1)))$ is a Z -set in M .

Since N_0 is a Z -set Q -manifold in M_0 there is an open embedding $\varphi'': N_0 \times [0, 2) \rightarrow M_0$ such that $\varphi''(n_0, 0) = n_0$ for every $n_0 \in N_0$. Now, $i_1 = \text{id}_N \cup (\varphi'|_{N_0 \times [0, 1)})$ and $i_2 = \text{id}_N \cup (\varphi''|_{N_0 \times [0, 1)})$ are two embeddings of $N \cup (N_0 \times [0, 1))$ onto Z -sets in M that are homotopic (both are homotopic to id_N ; simply collapse collars onto $N_0 \times \{0\} \subset N$). By Z -set unknotting, there is $G: M \rightarrow M$ such that $G \circ i_1 = i_2$. Then $G \circ \varphi': N \times [0, 1) \rightarrow M$ is our φ . \square

Using Lemmas 2.2 - 2.6, we can now prove the other half of Theorem 2.1 following step by step Chapman-Siebenmann's proof in [17].

For the convenience of the reader we will outline this proof in some detail since rather non-trivial technical adjustment must be performed.

Suppose that a Q -manifold pair, (M, M_0) , admits a boundary (B, B_0) and let (N, N_0) be a compact Q -manifold pair that compactifies (M, M_0) . With the help of Lemmas 2.3 and 2.5 (applied to the projection $(N, N_0) \times I \rightarrow (N, N_0)$) we may replace (N, N_0) by $(N, N_0) \times I$ and assume that $(B, B_0) \subset (N, N_0) \times \{0\}$. With this notation, we must prove that $(N, N_0) \times [0, 1] - (B, B_0)$ is homeomorphic to some $\text{Map}(\sigma, \sigma_0) \times Q$.

Since (N, N_0) can be triangulated (Lemma 2.2), we can write $(B, B_0) = \bigcap_{i=1}^{\infty} (M_i, M_{0i})$, where

$$(N, N_0) \times \{0\} \supset (M_1, M_{01}) \supset (M_2, M_{02}) \supset \dots$$

is a basis of compact Q -manifold pair neighborhoods of (B, B_0) in $(N, N_0) \times \{0\}$. Let $\varphi_i : M_i \times [0, 1] \rightarrow N \times [0, 1]$ be a collar on M_i such that $\varphi_i(M_{0i} \times [0, 1]) \subset N_0 \times [0, 1]$. If the image of φ_i is close to M_i , then $(N_i, N_{0i}) = (\varphi_i(M_i \times [0, 1/2]), \varphi_i(M_{0i} \times [0, 1/2]))$ will be a decreasing sequence of compact Q -manifold pair neighborhoods of (B, B_0) in $(N, N_0) \times I$ such that

$$(B, B_0) = \bigcap_{i=1}^{\infty} (N_i, N_{0i}) ,$$

each $(N_i, N_{0i}) \cong (\delta N_i, \delta N_{0i}) \times [0, 1]$, and $(\delta N_i, \delta N_{0i})$ is a compact Q-manifold pair collared in both (N_i, N_{0i}) and $(N, N_0) \times I - \text{int}(N_i, N_{0i})$.

Choose compact polyhedral pairs (X'_0, X'_{00}) , (X'_1, X'_{01}) , ... and homeomorphisms

$$h'_0: (X'_0, X'_{00}) \times Q \rightarrow (N, N_0) \times [0, 1] - \text{int}(N_1, N_{01})$$

$$h'_1: (X'_1, X'_{01}) \times Q \rightarrow \delta(N_1, N_{01})$$

$$h'_2: (X'_2, X'_{02}) \times Q \rightarrow \delta(N_2, N_{02})$$

⋮
⋮
⋮

We will construct polyhedral pairs (X_i, X_{0i}) and maps $f_i: (X_{i+1}, X_{0i+1}) \rightarrow (X_i, X_{0i})$ such that if $(\sigma, \sigma_0) = \{X_i, X_{0i}, f_i\}$ then $\text{Map}(\sigma, \sigma_0) \times Q \cong (N, N_0) \times [0, 1] - (B, B_0)$.

First, define $f'_0: (X'_1, X'_{01}) \rightarrow (X'_0, X'_{00})$ and $f'_i: (X'_{i+1}, X'_{0i+1}) \rightarrow (X'_i, X'_{0i})$ ($i > 0$) so that the following rectangles commute up to homotopy.

$$\begin{array}{ccc} & h_1 & \\ (X'_1, X'_{01}) \times Q \rightarrow \delta(N_1, N_{01}) & & \\ \downarrow f'_0 \times \text{id} \quad h_0 & \searrow & \\ (X'_0, X'_{00}) \times Q \rightarrow (N, N_0) \times [0, 1] - \text{int}(N_1, N_{01}) & & \\ & & \\ (X'_{i+1}, X'_{0i+1}) \times Q & \xrightarrow{h_{i+1}} & \delta(N_{i+1}, N_{0i+1}) \\ \downarrow f'_i \times \text{id} & & \downarrow \kappa_i \\ (X'_i, X'_{0i}) \times Q & \xrightarrow{h_i} & \delta(N_i, N_{0i}) . \end{array}$$

Here κ_i is a composition $\delta(N_{i+1}, N_{0i+1}) \hookrightarrow (N_i, N_{0i}) - \text{int}(N_{i+1}, N_{0i+1}) \xrightarrow{\lambda_i} \delta(N_i, N_{0i})$ where λ_i is a homotopy inverse of the inclusion $\delta(N_i, N_{0i}) \hookrightarrow (N_i, N_{0i}) - \text{int}(N_{i+1}, N_{0i+1})$. This inclusion is in fact relatively homotopic to a homeomorphism as is seen from the commutative diagram below using Lemma 2.5.

$$\begin{array}{ccc}
 \delta(N_i, N_{0i}) & \hookrightarrow & (N_i, N_{0i}) - \text{int}(N_{i+1}, N_{0i+1}) \\
 \alpha \swarrow & & \nearrow \beta \\
 & (N_i, N_{0i}) & .
 \end{array}$$

The maps α and β are natural collapses of closed collars onto bases and thus are CE-maps.

Let X_i be the mapping cylinder of $X'_{0i} \hookrightarrow X'_i$ and let X_{0i} be the top of X_i . Let $f_i: (X_{i+1}, X_{0i+1}) \rightarrow (X_i, X_{0i})$ be defined level-wise as f'_i . This change has been made to ensure that $\text{Map}(X_{0i}, f_i|_{X_{0i+1}})$ is a Z-set in $\text{Map}(X_i, f_i)$.

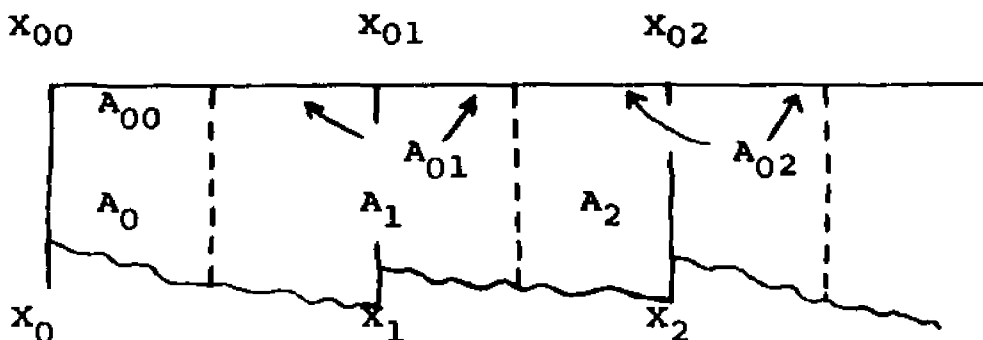
Observe that for each i , the diagram

$$\begin{array}{ccc}
 (X_{i+1}, X_{0i+1}) & \xrightarrow{c_{i+1}} & (X'_{i+1}, X'_{0i+1}) \\
 \downarrow f_i & & \downarrow f'_i \\
 (X_i, X_{0i}) & \xrightarrow{c_i} & (X'_i, X'_{0i})
 \end{array}$$

commutes (c_i 's are collapses of mapping cylinders onto bases) and, by Lemma 2.5, c_i is relatively homotopic to a homeomorphism $h'_i: (X_i, X_{0i}) \times Q \rightarrow (X'_i, X'_{0i}) \times Q$ (note that $c_i|_{X_{0i}}: X_{0i} \rightarrow X'_{0i}$ is a CE-map so that proof of Lemma 2.5 goes through without the assumption $f^{-1}(N_0) = M_0$). Now, define $h_i = h'_i \circ h_i'$ for each $i \geq 0$. Then all rectangles commute up to homotopy if we replace primed letters with those without primes.

The rest of the proof is essentially as in [17]. We repeat the key steps here.

Consider the relative infinite mapping cylinder $M(\sigma, \sigma_0)$ and write $\text{Map}(\sigma) = A_0 \cup A_1 \cup \dots$ and $\text{Map}(\sigma_0) = A_{00} \cup A_{01} \cup \dots$ as in the picture



Observe that the added $[0,1]$ direction in each X_i makes each A_{0i} a Z -set in A_i .

The inclusion $(X_0, X_{00}) \times Q \hookrightarrow (A_0, A_{00}) \times Q$ is rela-

tively homotopic to a homeomorphism because its relative homotopy inverse, a collapse $(A_0, A_{00}) \times Q \rightarrow (X_0, X_{00}) \times Q$, is homotopic to a homeomorphism by Lemma 2.5. Using h_0 we can get a homeomorphism $g_0: (A_0, A_{00}) \times Q \rightarrow (N, N_0) \times [0, 1] - \text{int}(N_1, N_{01})$ which by Lemma 2.3 can be adjusted so that $g_0|_{X_1 \times \{1/2\} \times Q}$ is given by h_1 . In a similar way, we define

$$g_i: (A_i, A_{0i}) \times Q \rightarrow (N_i, N_{0i}) - \text{int}(N_{i+1}, N_{0i+1})$$

that match nicely. \square

We close this chapter by stating Felt's generalization [23] of Theorem 1.4.

Theorem 2.7. Two Z -set pairs (X, X_0) and (Y, Y_0) in (Q, W) have the same shape in the sense of Mardesić and Segal if and only if $(Q-X, W-X_0)$ and $(Q-Y, W-Y_0)$ are homeomorphic as pairs.

CHAPTER III

SHAPE DIMENSION

In this chapter, we shall prove several properties of shape dimension. Our proofs use the absolute version of Theorem 2.1.

Let $n \geq 0$ be an integer. A locally compact separable metric space M is tame- (n-tame) at $-\infty$ provided, that for every compactum $A \subset M$, there is a compactum $B \supset A$ and a (an n-dimensional) finite complex K such that the inclusion $M-B \hookrightarrow M-A$ factors up to homotopy through K . The notion of tame-at- $-\infty$ space is due to Chapman and Siebenmann [17].

Lemma 3.1. Let M and N be spaces such that N homotopy dominates M at ∞ . If N is tame- (n-tame) at- ∞ , then so is M .

Proof. Let $f:N \rightarrow M$ and $g:M \rightarrow N$ be proper maps such that $f \circ g$ is homotopic-at- ∞ to id_M ; i.e., for any compactum $A \subset M$, there is a compactum $A' \supset A$ and a homotopy $H:(M-A') \times I \rightarrow M-A$ such that H_0 is the inclusion

$M-A' \hookrightarrow M-A$ and $H_1 = f \circ g|_{M-A'}$. Let $B' \supset f^{-1}(A')$ be a compactum in N for which there is a (an n -dimensional) complex K such that for some maps α', β'

$$\begin{array}{ccc} N-B' & \hookrightarrow & N-f^{-1}(A') \\ \alpha' \searrow & & \nearrow \beta' \\ & K & \end{array}$$

homotopy commutes. Put $B = g^{-1}(B')$, $\alpha = \alpha' \circ (g|_{M-B})$, and $\beta = (f|_{N-f^{-1}(A')}) \circ \beta'$. Clearly, the diagram

$$\begin{array}{ccc} M-B & \hookrightarrow & M-A \\ \alpha \searrow & & \nearrow \beta \\ & K & \end{array}$$

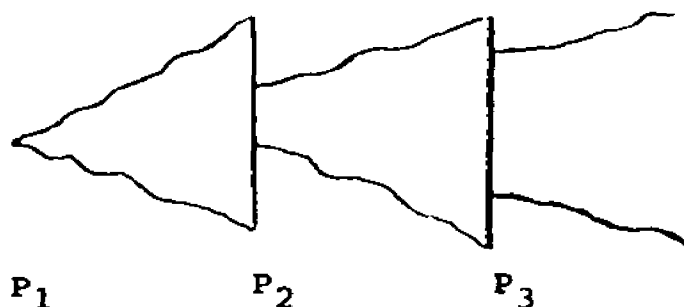
homotopy commutes, i.e., M is tame- (n -tame) at- ∞ . [1]

A shape dimension $Sd(X)$ of a compactum X is defined as $\min\{\dim Y | Sh(Y) \geq Sh(X)\}$, where $Sh(Y) \geq Sh(X)$ means that Y shape dominates X , a notion first introduced by Borsuk [5].

Our next theorem gives a new characterization of the inequality $Sd(X) \leq n$ in terms of homotopy properties of the complement $M = Q-X$ where X is considered as a Z -set in the Hilbert cube Q .

Theorem 3.2. For a Z -set X in Q , $Sd(X) \leq n$ if and only if $M = Q - X$ is n -tame-at- ∞ .

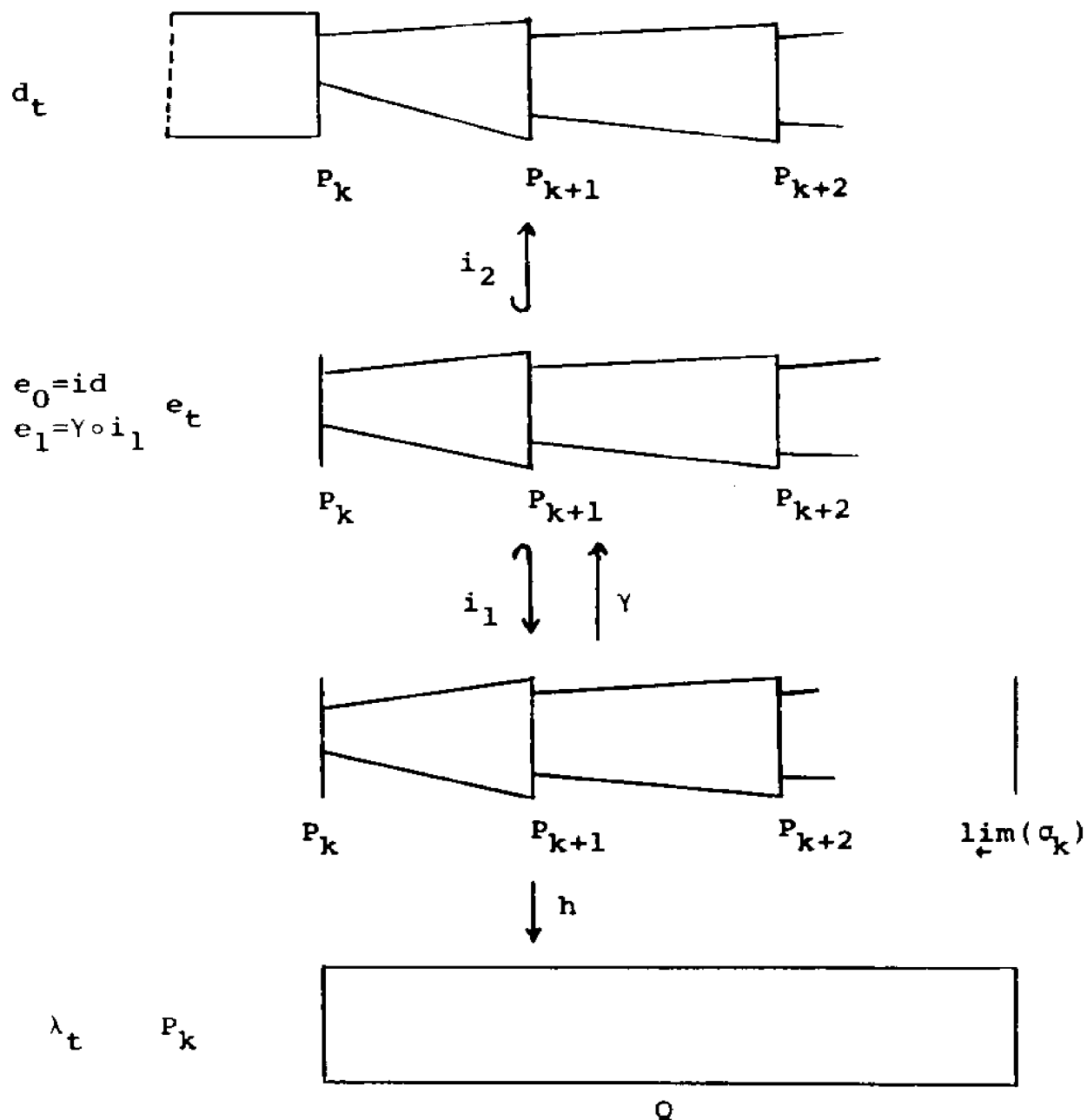
Proof. Suppose X is a Z -set in Q and $Sd(X) \leq n$. Then there is an n -dimensional compactum Y such that $Sh(Y) \geq Sh(X)$. We can represent Y as the inverse limit $\varprojlim \sigma$ of an inverse sequence $\sigma = \{P_i, f_i\}$ where $P_1 =$ point and $\dim P_i \leq n$, for every $i > 0$. The infinite mapping cylinder $Map(\sigma) \times Q$ pictured below



when compactified by adding $(\varprojlim \sigma) \times Q$ is homeomorphic to Q by the Cylinder Completion Theorem 1.5 and $(\varprojlim \sigma) \times Q$ is a Z -set in $(Map(\sigma) \cup \varprojlim \sigma) \times Q \cong Q$. By Theorem 1.2, since $Sh(Y) = Sh(\varprojlim \sigma \times Q) \geq Sh(X)$, $Map(\sigma) \times Q$ weak proper homotopy dominates M . Hence, if we prove that $Map(\sigma) \times Q$ is n -tame-at- ∞ it would follow from Lemma 3.1 that M is n -tame-at- ∞ .

Let $A \subset Map(\sigma) \times Q$ be an arbitrary compactum. Pick k large enough so that $A \subset Map(\{P_1 \leftarrow P_2 \leftarrow \dots \leftarrow P_{k-1}\}) \times Q = B$. We will prove that there is a homotopy $g_t: V \rightarrow V$, with

$v = \text{Map}(\sigma) \times Q - B$, such that $g_0 = \text{id}$ and $\dim g_1(V) \leq n$.



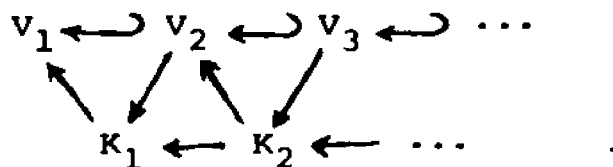
Let d_t be the strong deformation retraction of v onto $\text{Map}_k(\sigma) \times Q$ that slides $p_k \times [0,1)$ onto $p_k \times \{0\}$. In the picture, γ is a homotopy inverse of i_1 . It

exists since $\lim_{\leftarrow} \{P_k \leftarrow P_{k+1} \leftarrow \dots\} \times Q$ is a Z-set in $[\text{Map}_k(\sigma) \cup \lim_{\leftarrow} (\{P_k \leftarrow P_{k+1} \leftarrow \dots\})] \times Q$. By e_t we denoted a homotopy connecting id and $\gamma \circ i_1$. The map h is a homeomorphism of the latter space onto $P_k \times Q$ (again, we used the Cylinder Completion Theorem 1.5) while λ_t is a map $\lambda_t(x, q) = (x, (1-t)q)$. Then our g_t is defined by

$$g_t = \begin{cases} d_{3t} & , \quad 0 \leq t \leq 1/3 \\ i_2 \circ e_3(t-1/3) \circ d_1 & , \quad 1/3 \leq t \leq 2/3 \\ i_2 \circ \gamma \circ h^{-1} \circ \lambda_3(t-2/3) \circ h \circ i_1 \circ d_1 & , \quad 2/3 \leq t \leq 1 \end{cases} .$$

Observe that by the Mapping Replacement Theorem 1.1 we can assume γ is an embedding so that condition $\dim g_1(V) \leq n$ indeed holds.

Conversely, suppose a Z-set X has the property that $M = Q - X$ is n -tame-at- ∞ . Then we can find a sequence $M = V_1 \supset M = V_2 \supset V_3 \supset V_4 \supset \dots$ of open subsets of M with compact complements and $\bigcap_{i>0} V_i = \emptyset$ such that for some sequence $K_1 = \text{point}$, K_2, K_3, K_3, \dots , where K_2, K_3, \dots are n -dimensional finite polyhedra we can form a commutative diagram



In Appendix II of [17] it was shown that under these

conditions there is a proper homotopy equivalence

$\text{Map}(\sigma) \times Q \rightarrow M$, where $\sigma = \{K_1 \leftarrow K_2 \leftarrow \dots\}$. Since both M and $\text{Map}(\sigma) \times Q$ are contractible Q -manifolds admitting boundaries (as defined in [17], see §2) by [17, Theorems 7 and 9], $M \cong \text{Map}(\sigma) \times Q$ and $\text{Sh}(\varprojlim \sigma \times Q) = \text{Sh}(\varprojlim \sigma) = \text{Sh}(X)$. But, $\dim(\varprojlim \sigma) \leq n$ so that $\text{Sd}(X) \leq n$. \square

Note. It is clear from the above proof that a Z -set X in Q has a shape dimension $\leq n$ if and only if X has arbitrary small Q -manifold nbds of the form $K \times Q$ where K is an at most n -dimensional finite complex. Therefore, the intersection X of Z -sets $X_1 \supset X_2 \supset \dots$ in Q with $\text{Sd}(X_k) \leq n$ for every $k = 1, 2, \dots$ also satisfies $\text{Sd}(X) \leq n$ ([39, Theorem (3.1)]).

Corollary 3.3. If $\text{Sd}(X) = n$ then there is an n -dimensional compactum Y such that $\text{Sh}(X) = \text{Sh}(Y)$.

Proof. Consider X as a Z -set in Q . The equality $\text{Sd}(X) = n$ by the above theorem implies that $M = Q - X$ is n -tame-at- ∞ . With the notation from the second half of the proof for Theorem 3.2, since $\dim K_i = n$ for every $i > 1$, $\dim(\varprojlim \sigma) \leq n$. Then $Y = \varprojlim \sigma$ is the required compactum because $\dim Y < n$ is impossible. \square

Corollary 3.3 was earlier proved by Holsztyński (unpublished) and Nowak [39]. Also included in the proof of Theorem 3.2 is the following version of a result of Nowak

[39] with part (b) representing an improvement of his conclusion.

Corollary 3.4. Let $X \subseteq Q$ be a Z -set. (a) If for every neighborhood U of X in Q there is a homotopy $\varphi: X \times I \rightarrow U$ such that $\varphi_0 =$ the inclusion $X \hookrightarrow U$ and $\dim \varphi_1(X) \leq n$, then $Sd(X) \leq n$. (b) If $Sd(X) \leq n$, then for every neighborhood U of X in Q , there is a smaller neighborhood V and a deformation $\varphi: V \times I \rightarrow V$ with $\dim \varphi_1(V) \leq n$.

Proof. (a) The method in [7] can be applied to get a closed prism neighborhood $V \subseteq U$ of X and a homotopy $\psi: V \times I \rightarrow U$ such that $\psi_0 =$ the inclusion $V \hookrightarrow U$ and $\dim \psi_1(V) \leq n$. We claim that there is no loss of generality to assume $\psi_1(V)$ is an at most n -dimensional finite complex K . Indeed, let $\epsilon > 0$ be such that any map $f: \psi_1(V) \rightarrow U$ ϵ -close to the inclusion $i: \psi_1(V) \hookrightarrow U$ is homotopic in U with i . Since $\dim \psi_1(V) \leq n$ there is an at most n -dimensional finite complex K and an ϵ -map g of $\psi_1(V)$ onto K . By [31], we can find an embedding $h: K \rightarrow U$ with $d(h \circ g, i) < \epsilon$. The choice of ϵ gives us a homotopy $\chi: \psi_1(V) \times I \rightarrow U$ with $\chi_0 = i$ and $\chi_1(\psi_1(V)) = h \circ g \circ \psi_1(V)$. Then

$$\tilde{\psi}_t = \begin{cases} \varphi_2^t & , \quad 0 \leq t \leq 1/2 \\ \chi_{2t-1} \circ \psi_1 & , \quad 1/2 \leq t \leq 1 . \end{cases}$$

will fulfill our requirement.

To prove (a), it suffices to see that $M = Q - X$ is n -tame-at- ∞ . Let $A \subset M$ be a compactum. The complement $U = Q - A$ is a neighborhood of X in Q . Select $V \subset U$ and $\psi: V \times I \rightarrow U$ as above. Then $Q - \text{int } V$ is a compactum in M and the inclusion $M - B = \text{int } V - X \hookrightarrow M - A = U - X$ factors through K . To see this, recall [12], that because X is a Z -set in Q , there is a homotopy $g_t: U \rightarrow U$ such that $g_0 = \text{id}_U$ and $g_t(U) \cap X = \emptyset$ for all $t > 0$.

$$\begin{array}{ccc} \text{int } V - X & \hookrightarrow & U - X \\ i_1 \downarrow & & \uparrow g_1 \\ V - X & \hookrightarrow & U \\ \psi_1 \searrow & & \nearrow j \\ & K & \end{array} .$$

If we put $f_t(v) = g_t \circ \psi_t \circ i_1(v)$ for every $v \in \text{int } V - X$, then $f_0 =$ the inclusion $\text{int } V - X \hookrightarrow U - X$, and $f_1 = (g_1 \circ j) \circ (\psi_1 \circ i_1)$.

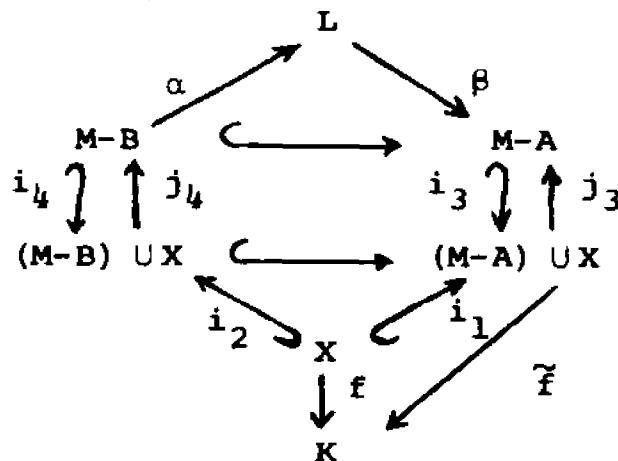
(b) was demonstrated in the first half of the proof for Theorem 3.2. \square

For a topological space X by a deformation dimension

$d \dim X$ we mean the smallest integer n such that every map $f: X \rightarrow K$ of X into a CW complex K is homotopic to a map $g: X \rightarrow K^n$ of X into the n -skeleton K^n of K . One can easily verify that the deformation dimension is a shape invariant. Here we give a new proof that, for compactum X , $d \dim X = \text{Sd}(X)$ (see [18, Corollary 2.4] and [39, Corollary (1.7)]).

Corollary 3.5. If X is a compact metric space, then $d \dim X = \text{Sd}(X)$.

Proof. (a) $d \dim X \leq \text{Sd}(X)$. Assume $\text{Sd}(X) = n$. We must show that, given an arbitrary map $f: X \rightarrow K$ of X into any CW-complex K , there is $g: X \rightarrow K^n$ with $f \simeq g$. Regard X as a Z -set in Q and select a compactum $A \subset M = Q - X$ such that there is an extension $\tilde{f}: (M - A) \cup X \rightarrow K$. By Theorem 3.2, we can find a compactum $B \supset A$ in M and an n -dimensional finite complex L such that the diagram below homotopy commutes.



Here, j_3 and j_4 are homotopy inverses of i_3 and i_4 , respectively, (they exist because X is a Z -set in Q). The composition $F = \tilde{f} \circ i_3 \circ \beta: L \rightarrow K$ is a map of an n -dimensional finite complex into K . It follows from the Cellular Approximation Theorem [32] that there is $G: L \rightarrow K^n$ with $F \simeq G$. Define $g: X \rightarrow K$ as $G \circ \alpha \circ j_4 \circ i_2$.

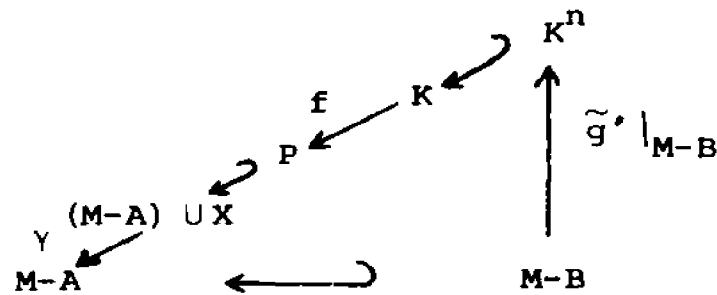
(b) $Sd(X) \leq d \dim X$. Assume $d \dim X = n$. Regarding X as a Z -set in Q , we will show that $M = Q - X$ is n -tame-at- ∞ , which suffices by Theorem 3.2.

Let $A \subset M = Q - X$ be an arbitrary compactum. Pick a closed prism neighborhood $P \subset (M - A) \cup X$ of X and let K be a finite complex, and $f: K \rightarrow P$, $g: P \rightarrow K$ maps such that $f \circ g \simeq id_P$. By assumption, $g|_X: X \rightarrow K$ is homotopic to a map $\tilde{g}: X \rightarrow K^n$. Let $\varphi: X \times I \rightarrow K$ be a homotopy connecting $g|_X$ and \tilde{g} and let $N \subset P$ be an open neighborhood of X for which there is an extension $\tilde{g}': N \rightarrow K^n$ of \tilde{g} . Then, on a closed subset $T = N \times \{0, 1\} \cup X \times [0, 1]$ of $N \times [0, 1]$ we can define a map ϕ into K by

$$\phi(x, t) = \begin{cases} g(x) & , \text{ if } t = 0 \text{ and } x \in N \\ \tilde{g}'(x) & , \text{ if } t = 1 \text{ and } x \in N \\ \varphi(x, t) & , \text{ if } t \in [0, 1] \text{ and } x \in X . \end{cases}$$

Since K is an ANR, there is a neighborhood T' of T in $N \times [0, 1]$ and an extension $\phi': T' \rightarrow K$ of ϕ . Let $B \subset M$ be a compactum with the property

$[(M-B) \cup X] \times [0,1] \subseteq T'$. Hence, $\tilde{g}'(M-B) \subseteq K^n$. Finally, if $\gamma: (M-A) \cup X \rightarrow M-A$ is a homotopy inverse of the inclusion $M-A \hookrightarrow (M-A) \cup X$, then the diagram



homotopy commutes, proving that M is n -tame-at- ∞ . \square

Our Theorem 3.2 can be conveniently formulated as follows. The advantage of this form is that questions about a shape dimension of a Z -set $X \subseteq Q$ are reduced to the questions about n -domination of the complement $Q-X$ in the proper category.

Recall that every Q -manifold M can be written as $P \times Q$, where P is a locally finite countable simplicial complex [13]. We will define a formal dimension for $-d(M)$ of M as $\min\{\dim P \mid P \text{ is a polyhedron and } P \times Q \cong M\}$.

Corollary 3.6. (a) If for a Z -set $X \subseteq Q$, $Sd(X) \leq n$, then $-d(Q-X) \leq n+1$. (b) Equality $-d(Q-X) = n+1$ implies $Sd(X) = n$.

Proof. If $Sd(X) = n$, then there is, by Corollary 3.3, an n -dimensional compactum Y such that $Sh(Y) = Sh(X)$.

We can represent Y as $\varprojlim \sigma$ where $\sigma = \{P_1 \leftarrow P_2 \leftarrow \cdots\}$ is an inverse sequence of compact polyhedra with $P_1 = \text{point}$ and $\dim P_i \leq n$, for all $i > 1$. But $\text{Map}(\sigma) \times Q \cong Q-X$ and, obviously, for $-d(\text{Map}(\sigma) \times Q) \leq n+1$.

On the other hand, if for $-d(Q-X) = n+1$, then the assumption $\text{Sd}(X) < n$ would lead to a contradiction since then $\text{Sh}(X) = \text{Sh}(\varprojlim \sigma)$, where σ is an inverse sequence of $(n-1)$ -dimensional polyhedra beginning with the point, and $Q-X \cong \text{Map}(\sigma) \times Q$ so that for $-d(Q-X) \leq (n-1) + 1 = n$. \square

Here is a method showing how to use the above result in studying the properties of shape dimension. Take an arbitrary polyhedron P for which $P \times Q \cong Q-X$. Suppose we established that P is proper homotopy dominated by an n -dimensional polyhedron P' . We claim that $P \times Q$ is n -tame-at- ∞ . Since P' proper homotopy dominates $P \times Q$ for any compactum $A \subset P \times Q$ there is a compactum $B \supset A$, a subpolyhedron P'' of P , and maps $f: P'' \rightarrow P \times Q - A$, $g: P \times Q - B \rightarrow P''$ such that

$$\begin{array}{ccc}
 & P'' & \\
 g \nearrow & & \searrow f \\
 P \times Q - B & \xhookrightarrow{\quad} & P \times Q - A
 \end{array}$$

homotopy commutes. But we can assume that B is a Q -manifold such that $\text{Bd } B$ is a collared Q -manifold both

in B and $P \times Q - \text{int } B$ so that $P \times Q - B$ is homotopy equivalent to a finite complex K , i.e., there are maps $\psi: P \times Q - B \rightarrow K$ and $\varphi: K \rightarrow P \times Q - B$ with $\varphi \circ \psi \simeq \text{id}_{P \times Q - B}$. Since $g \circ \varphi(K)$ is compact, there is a finite polyhedron P''' of P'' containing $g \circ \varphi(K)$. Then the factorization

$$\begin{array}{ccc}
 & P''' & \\
 g \circ \varphi \circ \psi \nearrow & & \searrow f|_{P'''} \\
 P \times Q - B & \hookrightarrow & P \times Q - A
 \end{array}$$

supports our claim.

S. Nowak [39] gave a rather complicated proof of the next proposition. The proof we present shows some merits of our approach to questions of shape.

Proposition 3.7. Let X_1, X_2 be compacta. Then $\text{Sd}(X_1 \cup X_2) \leq \max(\text{Sd}(X_1), \text{Sd}(X_2), \text{Sd}(X_1 \cap X_2) + 1)$.

Proof. Denote the integers appearing in the above inequality by k, l, m, n , respectively. Hence, we must prove $k \leq \max(l, m, n)$. Consider $X = X_1 \cup X_2$ as a Z -set in $Q \times [-1, 1]$ with $X_1 \subset Q \times [-1, 0]$, $X_2 \subset Q \times [0, 1]$, and $X \cap (Q \times \{0\}) = X_0 = X_1 \cap X_2$ being a Z -set in $Q \times \{0\}$. Since $\text{Sd}(X_1) = l$, $\text{Sd}(X_2) = m$, there are, by Theorem 3.2, arbitrary small closed Q -manifold neighborhoods N_1 of X_1 in $Q \times [-1, 0]$ and N_2 of X_2 in $Q \times [0, 1]$ such that

$N_1 \cong K_1 \times Q$, $N_2 \cong K_2 \times Q$, where K_1 and K_2 are finite complexes of dimensions l and m , respectively. Pick a closed Q -manifold neighborhood $N_0 \subseteq N_1 \cap N_2$ of X_0 in $Q \times \{0\}$ that is homeomorphic to $K_0 \times Q$ for some $(n-1)$ -dimensional finite complex K_0 . But N_0 is a Z -set submanifold in both N_1 and N_2 so that, by Chapman's Relative Triangulation Theorem (see Lemma 2.2), we can find complexes and homeomorphisms $h_1: N_1 \rightarrow K'_1 \times Q$, $h_2: N_2 \rightarrow K'_2 \times Q$ extending a fixed homeomorphism $h_0: N_0 \rightarrow K_0 \times Q$. Moreover, $\dim K'_1 = \max(l, n)$ and $\dim K'_2 = \max(m, n)$. Let $N \subseteq N_1 \cup N_2$ be any neighborhood of X for which $N \cap (Q \times \{0\}) = N_0$, and let K be the result of gluing K'_1 and K'_2 along K_0 . We claim that the inclusion $N-X \hookrightarrow (N_1 \cup N_2) - X$ factors through K . To see this, let $\lambda_t: Q \times [-1, 1] \rightarrow Q \times [-1, 1]$ move Q off of X with $\lambda_t|_{(Q \times [-1, 1]) - N} = \text{id}$, for all t . Define $\alpha: N-X \rightarrow K$ by $\alpha = p \circ h_N|_{N-X}$, where $h_N = (h_1|_{N \cap N_1}) \cup (h_2|_{N \cap N_2})$ and p is the projection $K \times Q \rightarrow K$. Also, $\beta: K \rightarrow (N_1 \cup N_2) - X$ is given by $\beta = \lambda_1 \circ h_N^{-1} \circ (\times 0)$. One can easily check that the diagram

$$\begin{array}{ccc}
 N-X & \hookrightarrow & (N_1 \cup N_2) - X \\
 \alpha \searrow & & \nearrow \beta \\
 & K &
 \end{array}$$

homotopy commutes, which proves that $(Q \times [-1, 1]) - X$ is

$\max(\ell, m, n)$ -tame-at- ∞ , i.e., $k \leq \max(\ell, m, n)$. \square

Remark 3.8. If $X_0 = X_1 \cap X_2$ has trivial shape, then we can take K_0 to be a point and then $\text{Sd}(X_1 \cup X_2) = \max(\text{Sd}(X_1), \text{Sd}(X_2))$ ([39, Theorem (4.19)]).

CHAPTER IV

MOVABILITY

In this chapter, we shall first characterize movability of a Z -set X in Q in terms of homotopy invariants at ∞ of $Q-X$ as introduced in Chapter I, and then apply this apparatus to get new theorems whose corollaries include new proofs (in most cases simplified) of standard results in shape theory of compacta.

Recall Borsuk's definition of movability. A compactum $X \subset Q$ is (Borsuk) movable provided that for every neighborhood U of X in Q , there is a smaller neighborhood V of X such that given an arbitrary neighborhood W of X , there is a homotopy $H: V \times I \rightarrow U$ with $H_0 =$ the inclusion $V \hookrightarrow U$ and $H_1(V) \subset W$.

Theorem 4.1. A Z -set $X \subset Q$ is (Borsuk) movable if and only if the Q -manifold $M = Q-X$ satisfies the following property:

(M) For every compactum $A \subset M$, there is a compactum $B \supset A$ such that, given any map $\varphi: P \rightarrow M-B$ of a finite polyhedron P into $M-B$ and any compactum $C \supset B$ there

is a map $\psi: P \rightarrow M-C$ with $\varphi \simeq \psi$ in $M-A$.

Proof. Assume a Z -set $X \subset Q$ is (Borsuk) movable. If $A \subset M = Q-X$ is an arbitrary compactum, $U = Q-A$ is a neighborhood of X in Q . Pick an open neighborhood $V \subset U$ as in the above definition, and set $B = Q-V$. Let $\varphi: P \rightarrow M-B = V-X$ be a map of a finite complex P into $V-X$ and let $C \supset B$ be a compactum. Let $H: V \times I \rightarrow U$ be a homotopy with $H_0 =$ the inclusion $V \hookrightarrow U$ and $H_1(V) \subset W = Q-C = (M-C) \cup X$. Also, let $G: V \times I \rightarrow V$ be a deformation such that $G_t(V) \subset V-X$ for all $t > 0$. It is clear that the map $\psi = G_1 \circ H_1 \circ \varphi: P \rightarrow M-C$ is homotopic in $M-A$ to a map $G_0 \circ H_0 \circ \varphi = \varphi$.

Conversely, suppose that condition (M) holds. Let U be an open neighborhood of X in Q . For a compactum $A = Q-U$ in M , select a compactum $B \supset A$ as in (M) and put $V' = Q-B$. Let $V \subset V'$ be a closed prism neighborhood of X in Q . We claim that this V does the job. Indeed, take any open neighborhood W of X in Q . We can always suppose $W \subset V'$. Then $C = Q-W$ is a compactum containing B . Observe that V can be written as $P \times Q_k$ where $P \subset I_1 \times \cdots \times I_{k-1}$ is a finite polyhedron. Let $\lambda_t: V \rightarrow V$ be a deformation for which $\lambda_t(V) \subset V-X$ for all $t > 0$. A map $\varphi = \lambda_1 \circ (\chi_0): P \rightarrow V-X \subset M-B$, where $\chi_0: P \rightarrow P \times Q_k = V$ is given by $\chi_0(p) = (p, 0)$, is homotopic via $\chi_t: P \rightarrow M-A = U-X$ to a map $\psi = \chi_1: P \rightarrow M-C = W-X$. Then a

homotopy $\xi_t: V \rightarrow U$ defined as

$$\xi_t(p, q_k) = \begin{cases} \lambda_{2t}(p, (1-2t)q_k) & , \quad 0 \leq t \leq 1/2 \\ \chi_{2t-1}(p) & , \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

satisfies $\xi_0 = \text{the inclusion } V \hookrightarrow U$ and $\xi_1(V) \subset W$. \square

Definition 4.2. A non-compact locally compact separable metric space M is called movable if M has property (\mathcal{M}) .

Corollary 4.3. A Z -set X in a compact Q -manifold N is (Borsuk) movable if and only if $M = N - X$ is movable.

Proof. We can assume X is contained in $N \times \{0\} \subset \subset N \times [0,1]$. Consider $N \times \{1\}$ as a Z -set in Q and let Q' denote the union of $N \times [0,1]$ and Q along $N \times \{1\}$. Then $Q' \cong Q$ and $N \times [0,1] - X$ is movable if and only if $Q' - X$ is movable. Now, Proposition 4.1 applies. \square

Theorem 4.4. Let M and N be non-compact locally compact separable metric spaces. If M is movable and M homotopy dominates N at ∞ , then N is also movable.

Proof. Let $f: M \rightarrow N$ and $g: N \rightarrow M$ be proper maps such that $f \circ g$ is homotopic-at- ∞ to id_N . If A is an arbitrary compactum in N , there is a compactum $A^* \supset A$ and a homotopy $H: (N - A^*) \times I \rightarrow N - A$ between $\text{id}_{N - A^*}$ and $f \circ g|_{N - A^*}$. Since M is movable, for a compactum $A' = f^{-1}(A^*)$ in M

we can select $B' \supset A'$ such that (M) holds. Let $B = g^{-1}(B') \cup A$. Assume $\varphi: P \rightarrow N-B$ is a map of a finite complex into $N-B$. For a composition $\varphi' = g \circ \varphi: P \rightarrow M-B'$ and a compactum $C' = f^{-1}(C) \cup B'$, where C is any compactum in N containing B , by assumption, there is a map $\psi': P \rightarrow M-C'$ homotopic to φ' in $M-A'$. But, $\psi = f \circ \psi': P \rightarrow N-C$ is then homotopic in $N-A^*$ to a map $f \circ g \circ \varphi$ and thus to φ in $N-A$. \square

Corollary 4.5 ([7]). If X is a (Borsuk) movable compactum and X shape dominates compactum Y , then Y is also (Borsuk) movable.

Proof. Regard X and Y as Z -sets in Q . It follows from Theorem 4.1 that $M = Q-X$ is movable and from Theorem 1.2 that M homotopy dominates $N = Q-Y$ at ∞ . Hence, Y is (Borsuk) movable by Theorem 4.4 and Theorem 4.1. \square

Definition 4.6. A locally compact separable metric space X is in the class $ANR(\infty)$, notation: $X \in ANR(\infty)$, if there is a compact ANR , Y , and a Z -set ANR , $Z \subset Y$, such that $Y-Z \cong X$.

Examples 4.7. (a) By the main result in [24] a non-compact Q -manifold M is in $ANR(\infty)$ if and only if M admits a Q -manifold boundary. This class of Q -manifolds

was completely identified in algebraic terms in [16].

(b) Sher [41] introduced the concept "docile-at- ∞ " for a locally compact $X \in \text{ANR}(\mathcal{M})$. His main result characterizes docile-at- ∞ spaces X as those for which the Freudenthal compactification FX of X is an ANR and the space EX of ends of X is a Z -set in FX . A similar result was established for so called contractible-at- ∞ spaces by Kozłowski [29]. In our terminology, every docile-at- ∞ space with finitely many ends is in $\text{ANR}(\infty)$. Since every 0-dimensional compactum is (Borsuk) movable, spaces that are docile-at- ∞ are in $\text{BM}(\infty)$ (see note following Corollary 4.8).

Corollary 4.8. If $X \in \text{ANR}(\infty)$, then X is movable.

Proof. We will present two proofs. The second is more elementary.

First proof. Let Y be a compact ANR and $Z \subset Y$ a Z -set ANR such that $X \cong Y-Z$. By [21], $Y \times Q$ is a compact Q -manifold so that $X \times Q$ admits a (Borsuk) movable boundary $Z \times Q$. Hence, by Corollary 4.3, $X \times Q$ is movable. Now, Theorem 4.4 implies that X is movable.

Second proof. Let $A \subset X' = Y-Z$ be a compactum. Since Z is an ANR, there is a neighborhood $U \subset Y-A$ of Z in Y and a retraction $r: U \rightarrow Z$. Y being an ANR

implies the existence of $\epsilon > 0$ such that every two ϵ -close maps into U are homotopic in $Y-A$ (we might need to pass to a smaller U). A standard argument by contradiction shows that there is a small open neighborhood $U_\epsilon \subset U$ of Z in Y such that the inclusion $i:U_\epsilon \hookrightarrow U$ and $r|_{U_\epsilon}$ are ϵ -close maps and therefore homotopic in $Y-A$ via h_t . A compactum B that we are looking for is $Y-U_\epsilon$. Indeed, suppose a map $\varphi:P \rightarrow X-B = U_\epsilon - Z$ of a finite polyhedron P into $X'-B$ and a compactum $C \supset B$ in X' are given. Since Z is a Z -set in Y there is a deformation $\lambda_t:Y-C \rightarrow Y-C$ such that $\lambda_t(Y-C) \subset X'-C$ for all $t > 0$. Then $\psi = \lambda_1 \circ r \circ \varphi:P \rightarrow X'-C$ is homotopic in $X'-A$ to φ via a homotopy $\lambda_t \circ h_t$. \square

Note. It is clear from the first proof that a non-compact locally compact ANR, X , is movable if $X \in BM(\infty)$, i.e., if there exists a compact ANR, Y , and a (Borsuk) movable Z -set $Z \subset Y$ such that $X \approx Y-Z$.

By restricting the dimension of a polyhedron P in (M) to be $\leq n$, we will get the concept of an n -movable locally compact space. It is easy to verify that results 4.1 - 4.5 hold when movable is replaced by n -movable and (Borsuk) movable with (Borsuk) n -movable. (This notion was studied in [8], [30].)

Theorem 4.9. If M is an n -movable and n -tame-at- (∞)

locally compact separable metric space, then M is movable.

Proof. Let A be a compactum in M . Pick a compactum $B' \supset A$ such that (\mathcal{M}) holds for all finite polyhedra P of dimension $\leq n$. Using n -tameness of M at ∞ , there is a compactum $B \supset B'$ and an n -dimensional finite polyhedron K such that the diagram

$$\begin{array}{ccc} M-B & \xrightarrow{\quad} & N-B' \\ \alpha \searrow & & \nearrow \beta \\ & K & \end{array}$$

homotopy commutes. Then B is the required compactum.

Indeed, let $\varphi: P \rightarrow M-B$ be an arbitrary map of a finite complex P into $M-B$ and $C \supset B$ any compactum. By assumption, there is a map $\beta': K \rightarrow M-C$ with $\beta' \simeq \beta$ in $M-A$ because $\dim K = n$. The map $\psi = \beta' \circ \alpha \circ \varphi: P \rightarrow M-C$ is in $M-A$ homotopic to $\beta \circ \alpha \circ \varphi$ and thus to φ . \square

Theorem 4.9 trivially implies the following: Every (Borsuk) n -movable compactum with a shape dimension $\leq n$ is (Borsuk) movable ([8], [30]).

In [34], Mardesić proved that an n -dimensional LC^{n-1} compactum is movable. Later on, simpler proofs of this theorem were presented in [40] and [8]. Applying an elementary lemma from [28] we are able to give a generalization

in the following form.

Theorem 4.10. If X is an LC^{n-1} compactum of a shape dimension $\leq n$, then X is movable.

Proof. By Theorem 4.9, it suffices to prove that the complement $M = Q - X$ is n -movable, where X is considered as a Z -set in Q . Since X is LC^{n-1} , the inclusion $i: X \hookrightarrow Q$ is a strong local connection in dimension $n-1$ (see [28] for definitions and notation). Let $A \subset M$ be an arbitrary compactum and $U \subset Q - A$ an open neighborhood of X such that $\mathcal{U} = \{Q - cl U, Q - A\}$ is an open cover of Q . Pick $\mathcal{W} < \mathcal{U}$ such that \mathcal{W} -close maps into Q are \mathcal{U} -homotopic. Let \mathcal{V} be a refinement of \mathcal{W} for which the assertion $E(\mathcal{V}, \mathcal{W}, n)$ holds ([28, Lemma 1]). In other words, given an at most n -dimensional finite simplicial complex K and a map $h: K \rightarrow Q$ that maps every closed simplex σ of K into some member of the collection $\{V \in \mathcal{V} \mid V \cap X \neq \emptyset\}$, there is $h': K \rightarrow X$ such that for every simplex σ of K we can find $W \in \mathcal{W}$ with $i \circ h'(\sigma) \cup h(\sigma) \subset W$. The choice of \mathcal{W} assures that, then h and h' are homotopic in $Q - A$. The compactum $B = Q - \bigcup \{V \in \mathcal{V} \mid V \cap X \neq \emptyset\}$ is the one we are looking for. Indeed, if C is a compactum in M containing B , let $\lambda_t: Q \rightarrow Q$ be a deformation with $\lambda_t(Q) \subset Q - X$, $\lambda_t(Q - A) \subset Q - A$, and $\lambda_t(Q - C) \subset Q - C$ for all $t > 0$. Suppose $h: K \rightarrow M - B \subset \bigcup \{V \in \mathcal{V} \mid V \cap X = \emptyset\}$ is any map of an at most

n -dimensional finite simplicial complex K into $M-B$. We can assume by the previous discussion that there is a map $h':K \rightarrow X$ homotopic in $M-A$ to h . Let $\varphi_t:K \rightarrow M-A$ be a homotopy in $Q-A$ connecting h and $\lambda_t \circ h':K \rightarrow M-C$. Then $\lambda_{t(1-t)} \varphi_t$ is a homotopy in $M-A$ between h and $\lambda_1 \circ h'$. Hence, M is n -movable. \square

Theorem 4.11. If M is a movable locally compact separable metric space, then $M \times (0,1)$ and $M \times [0,1)$ are movable.

Proof. Let $A \subset M \times (0,1)$ be an arbitrary compactum. Let $B' \supset \pi_M(A)$ be selected so that for every map $f:P \rightarrow M-B'$ of a finite complex P into $M-B'$ and a compactum $C' \supset B'$ there is a map $g:P \rightarrow M-C'$ with $f \simeq g$ in $M-\pi_M(A)$. Then $B = B' \times B''$, where B'' is a compact subinterval containing $\pi_{(0,1)}(A)$, is a compactum that works. Indeed, let $\varphi:P \rightarrow M \times (0,1) - B$ be a map of a finite complex P and let $C \supset B$ be any compactum. Open sets $V_1 = (M-B') \times (0,1)$, $V_2 = M \times ((0,1) - B'')$ and $V_0 = (M-B') \times ((0,1) - B'')$, cover $M \times (0,1) - B$. In case $\varphi(P)$ is contained completely either in V_1 or V_2 , we can easily homotop φ off of C by changing just one coordinate. Hence, we can assume that $\varphi(P)$ intersects each set V_0 , $V_1 - V_0$, $V_2 - V_0$. Then there is a fine subdivision P' of P such that for every simplex σ of P' some member of $\{V_0, V_1, V_2\}$ contains

$\varphi(\sigma)$. By P'_i ($i = 0, 1, 2$), we will denote a subcomplex of P' consisting of all simplices σ for which $\varphi(\sigma) \subset V_i$. If P' is fine enough, $P'_1 - P'_0$ and $P'_2 - P'_0$ will be non-empty and P'_0 will have non-empty interior. Let $\rho: P \rightarrow [0, 1]$ be a continuous function, that is, 1 on $P'_1 - P'_0$ and is 0 on $P'_2 - P'_0$. Let $\lambda_t: (0, 1) \rightarrow B'' \rightarrow (0, 1) \rightarrow B''$ be a deformation with $\lambda_1((0, 1) \rightarrow B'') \subset \subset (0, 1) \rightarrow \pi_{(0, 1)}(C)$. A homotopy h_t on P' defined by $h_t(p) = (\pi_M \varphi(p), \lambda_t \cdot \rho(p) \pi_{(0, 1)} \varphi(p))$ when $p \in P'_2$ and $h_t(p) = \varphi(p)$ for $p \in P'_1 - P'_0$ stays in $M \times (0, 1) \rightarrow B$. Observe that $\pi_M \circ h_1|_{P'_1}$ maps P'_1 into $M \rightarrow B'$. Let $g_t: P'_1 \rightarrow M \rightarrow \pi_M(A)$ be a homotopy of $\pi_M \circ h_1|_{P'_1}$ to a map of P'_1 into $M \rightarrow \pi_M(C)$. Then $k_t: P \rightarrow M \times (0, 1) \rightarrow A$ defined as

$$k_t(p) = \begin{cases} (g_t(1-\rho(p))(p), \pi_{(0, 1)} h_1(p)) & , p \in P'_1 \\ h_1(p) & , p \in P'_2 - P'_1 \end{cases}$$

is a homotopy between h_1 and $k_1: P \rightarrow M \times (0, 1) \rightarrow C$ in $M \times (0, 1) \rightarrow A$.

The factor $[0, 1)$ also admits a deformation with the properties of λ_t and hence the same proof applies to this case. ()

Corollary 4.12 (Borsuk [9]). If X is a (Borsuk) movable compactum, then the suspension, $S(X)$, of X is also (Borsuk) movable.

Proof. Consider X a Z -set in Q . It is easy to see that $S(X) \subset S(Q)$ is a Z -set in $S(Q)$ and that $S(Q) \cong Q$ [27]. Hence, by Theorem 4.1, $S(X)$ is (Borsuk) movable provided $S(Q) - S(X) \cong (Q-X) \times (0,1)$ is movable. But this is the case, by Theorem 4.11, since $Q-X$ was assumed movable. \square

Definition 4.13. Let M be a locally compact separable metric space and $M_0 \subset M$ an open subset. We will say that M is isotopable into M_0 if for every compactum $B \subset M_0$ and every compactum $D \supset B$ in M there is an invertible isotopy $\lambda_t: M \rightarrow M$ such that $\lambda_1(D) \subset M_0$ and $\lambda_t|_B = \text{id}$, for all t .

Theorem 4.14. If a locally compact separable metric space M is the union of an increasing sequence of its movable open subsets M_i , $i \geq 0$, such that M_{i+1} is isotopable into M_i , for every $i > 0$, then M is movable.

Proof. Let $A \subset M$ be an arbitrary compactum. We can assume $A \subset M_0$. Since M_0 is movable, there is a compactum $B \supset A$ in M_0 such that for every map $\varphi': P \rightarrow M_0 - B$ of a finite polyhedron P into $M_0 - B$ and a compactum $C' \supset B$ in M_0 , there is $\psi': P \rightarrow M_0 - C'$ with $\varphi' \simeq \psi'$ in $M_0 - A$. Let $C \supset B$ be any compactum in M and $\varphi: P \rightarrow M - B$ a map. Pick an index k large enough so that $D = C \cup \varphi(P) \subset M_k$. For each $k \geq j > 0$, let $\lambda_t^j: M_j \rightarrow M_j$ be an invertible isotopy such that $\lambda_t^j|_B = \text{id}$ for all t

and all j , and $\lambda_1^k(D) \subset M_{k-1}$, $\lambda_1^{k-1}(\lambda_1^k(D)) \subset M_{k-2}, \dots$
 $\dots, \lambda_1^1 \circ \dots \circ \lambda_1^k(D) \subset M_0$. Now, define a homotopy $h_t: M_k \rightarrow M_k$
 by composing homotopies λ_t^j together. Then $h_0 = \text{id}$,
 $h_t(M_k - B) \subset M_k - B$ for all t , and $h_1(D) \subset M_0$. Let
 $g_t: P \rightarrow M_0 - A$ be a homotopy between $h_1 \circ \varphi$ and
 $g_1: P \rightarrow M_0 - h_1(C)$. Then the join of homotopies $h_t \circ \varphi$, g_t ,
 and $h_t^{-1} \circ g_1$ connects φ with the map into $M_k - C \subset M - C$. \square

We will say that a non-compact locally compact metric space M is flat if for any compactum $A \subset M$ every map $\varphi: P \rightarrow M$ of a finite polyhedron P into M is homotopic to a map $\psi: P \rightarrow M - A$.

Theorem 4.15. Let, for each $i > 0$, X_i be a compact metric space and $Y_i \subset X_i$ be a closed subset with $X_i - Y_i$ movable and flat. If $X = \prod_{i>0} X_i$ and $Y = \prod_{i>0} Y_i$, then $X - Y$ is movable.

Proof. Let A be a compactum in $X - Y$. Then $X - A$ is a neighborhood of Y in X . Pick an index n large enough so that $U_1 \times \dots \times U_n \times \prod_{j>n} X_j \subset X - A$, where U_i ($i = 1, \dots, n$) is an open neighborhood of Y_i in X_i . Since each $X_i - Y_i$ is movable, there are compacta $B_i \subset X_i - Y_i$ with $B_i \supset X_i - U_i$ and such that for any compactum $C_i \supset B_i$ and a map $\varphi_i: P \rightarrow (X_i - Y_i) - B_i$ of a finite polyhedron P , φ_i is homotopic inside $U_i - Y_i$ to a map into $(X_i - Y_i) - C_i$. Let $B = \bigcup_{i=1}^n (B_i \times \prod_{j \neq i} X_j)$.

Let $C \supset B$ be an arbitrary compactum in $X - Y$ and $\varphi: P \rightarrow (X - Y) - B$ any map of a finite complex P . Take small open neighborhoods V_i ($i = 1, \dots, k; k \geq n$) of Y_i in X_i such that $V_1 \times \dots \times V_k \times \prod_{i>k} X_i \subset X - C$. By the choice of sets B_i , each map $\varphi_i = \pi_i \circ \varphi: P \rightarrow (X_i - Y_i) - B_i$ obtained from φ by projecting onto X_i coordinate, is, for $i = 1, \dots, n$, homotopic in $U_i - Y_i$ to a map $\psi_i: P \rightarrow V_i - Y_i$, and for $i = n+1, \dots, k$, using flatness of $X_i - Y_i$, φ_i is homotopic to a map $\psi_i: P \rightarrow V_i - Y_i$. Hence, φ is homotopic outside A to a map $\psi_1 \times \dots \times \psi_k \times \varphi_{k+1} \times \varphi_{k+2} \times \dots: P \rightarrow (X - Y) - C$ proving that $X - Y$ is movable. \square

Corollary 4.16 [7]. The product $X = \prod_{i>0} X_i$ of countably many compacta is (Borsuk) movable if and only if each factor X_i is (Borsuk) movable.

Proof. Clearly $\text{Sh}(X) \geq \text{Sh}(X_i)$ for every $i > 0$, so that if X is (Borsuk) movable, by Corollary 4.5, then X_i has the same property.

Conversely, assume each X_i is (Borsuk) movable and is a Z -set in $Q_i \cong Q$. Applying Theorem 4.1 and Theorem 4.15, we see that $(\prod_{i>0} Q_i) - (\prod_{i>0} X_i)$ is movable. But $\prod_{i>0} Q_i \cong Q$ and $\prod_{i>0} X_i$ is a Z -set in $\prod_{i>0} Q_i$. Therefore, $X = \prod_{i>0} X_i$ is (Borsuk) movable. \square

An isolated end \in (see [4]) of a locally compact

metric space X will be called movable if for every neighborhood U of ϵ , there is another neighborhood $V \subset U$ of ϵ such that, for every neighborhood W of ϵ and a map $\varphi: P \rightarrow V \cap X$ of a finite polyhedron P into $V \cap X$, there is a homotopy $h_t: P \rightarrow U \cap X$ of φ with $h_1(P) \subset W \cap X$.

Theorem 4.17. Let X be a locally compact metric space with finitely many ends $\epsilon_1, \dots, \epsilon_n$. Then X is movable if and only if each end ϵ_i is movable.

Proof. Let $\epsilon_1, \dots, \epsilon_n$ be movable ends of X . Let $A \subset X$ be an arbitrary compactum. Then $(X-A) \cup \epsilon_i = U_i$ is a neighborhood of ϵ_i , for every $i = 1, \dots, n$. Pick disjoint neighborhoods V_1, \dots, V_n of $\epsilon_1, \dots, \epsilon_n$, respectively, as in the above definition. A compactum $B = X - \bigcup_{i=1}^n V_i$ is easily seen to satisfy condition (\mathcal{M}) .

Conversely, assume X satisfies condition (\mathcal{M}) . Let U_1 be a neighborhood of the end ϵ_1 of X . We can find disjoint neighborhoods U'_1, \dots, U'_n of $\epsilon_1, \dots, \epsilon_n$, respectively, with $U'_1 \subset U_1$ such that $A = X - \bigcup_{i=1}^n U'_i$ is a compactum in X . Take $B \supset A$ using movability of X . Then $V_1 = U'_1 - B$ is a neighborhood of ϵ_1 such that any map of a finite polyhedron into $V_1 \cap X$ deforms inside U_1 to a map arbitrarily close to ϵ_1 . Hence, the end ϵ_1 is movable. \square

Theorem 4.18. Let X_1 and X_2 be compacta of trivial shape and $Z = X_1 \cup X_2$ with $X_0 = X_1 \cap X_2$ (Borsuk) movable.

Then Z is also a (Borsuk) movable compactum.

Proof. Consider $K = Q \times [-1,1]$ and denote $K_1 = Q \times [-1,0]$, $K_0 = Q \times \{0\}$, and $K_2 = Q \times [0,1]$. Embed Z into K in the following way. First, embed X_0 as a Z -set in K_0 and K . Second, extend this embedding which can be done using the Mapping Replacement Theorem 1.1, to an embedding of X_1 onto a Z -set in K_1 , and finally, to an embedding of X_2 onto a Z -set in K_2 . Then Z will be a Z -set in K and we will prove that $M = K - Z$ is movable.

At this point it is important to observe that the complement of a (Borsuk) movable Z -set in Q satisfies (M) for any compactum in place of P because every closed subset of Q has arbitrary small prism neighborhoods.

Let $A \subset M$ be a compactum and put $A_i = K_i \cap A$ for $i = 0,1,2$. Let $B_1 \supset A_1$ and $B_2 \supset A_2$ be compacta in $K_1 - X_1$ and $K_2 - X_2$, respectively, such that $(K_i - X_i) - B_i$ is contractible ($i = 1,2$). Let $B_0 \supset K_0 \cap (B_1 \cup B_2)$ be a compactum in $K_0 - Z = K_0 - X_0$ that satisfies (M) with respect to X_0 , $(K_0 - X_0) \cap (B_1 \cup B_2)$ in $K_0 - X_0$. Define B as $B_0 \cup B_1 \cup B_2$.

Suppose $C \supset B$ is an arbitrary compactum in M and $\varphi: P \rightarrow M - B$ is a map of a finite polyhedron P into $M - B$. Select $D_1 \supset K_1 \cap C$ in $K_1 - X_1$ and $D_2 \supset K_2 \cap C$ in $K_2 - X_2$ so that $(K_i - X_i) - D_i$ ($i = 1,2$) are contractible.

For $i = 0, 1, 2$, let $P_i = \varphi^{-1}(K_i)$. By the above, note there is a homotopy $h_t: P_0 \rightarrow (K_0 - X_0) \cap (B_1 \cup B_2)$ with $h_0 = \varphi|_{P_0}$ and $h_1(P_0) \subset K_0 - (D_1 \cup D_2)$. Now, extend h_1 to a map $h_1^*: P_1 \rightarrow (K_1 - X_1) - D_1$ and to a map $h_1^{**}: P_2 \rightarrow (K_2 - X_2) - D_2$. Thus, we have maps of $P_1 \times \{0, 1\} \cup P_0 \times I$ into $(K_1 - X_1) - B_1$ and of $P_2 \times \{0, 1\} \cup P_0 \times I$ into $(K_2 - X_2) - B_2$. Extending each of them separately will give us the required homotopy of φ . \square

Corollary 4.19. If X is a compactum of trivial shape and $X_0 \subset X$ a (Borsuk) movable closed subset, then the space X/X_0 , obtained from X by shrinking X_0 to a point, is also (Borsuk) movable.

Proof. Embed X_0 as a Z -set in $Q \times \{0\}$ and extend this to an embedding of X onto a Z -set in $Q \times [0, 1]$ such that $X \cap (Q \times \{0\}) = X_0$. Let $C(Q) = Q \times [0, 1]/Q \times \{0\}$ and let $q: Q \times [0, 1] \rightarrow C(Q)$ be a natural projection. Then $C(Q)$ is homeomorphic to Q and $q(X)$ is a Z -set copy of X/X_0 in $C(Q)$. Also, $C(Q) - q(X) \cong (Q \times [0, 1]) - (Q \times \{0\} \cup X)$. But, by Theorem 4.18, $Q \times \{0\} \cup X$ is (Borsuk) movable, so that $C(Q) - q(X)$ is movable. \square

Example 4.20. Let $\sigma = \{X_i, f_i\}$ be an inverse sequence of finite polyhedra. Let $M^*(\sigma)$ denote the subspace $\text{Map}(\sigma) \times \{0\} \cup (\bigcup_{i>0} X_i \times [0, 1])$ of $\text{Map}(\sigma) \times [0, 1]$ where $X_i \subset \text{Map}(\sigma)$ is a copy of X_i in $\text{Map}(\sigma)$. Then $M^*(\sigma)$ is movable.

CHAPTER V

SHAPE ABSOLUTE NEIGHBORHOOD RETRACTS

The class of shape absolute neighborhood retracts (SANR's for short) was introduced by Borsuk [6] under the name of fundamental absolute neighborhood retracts (change in terminology was suggested by Kozłowski). It is easy to prove that it agrees with the class of all compacta that are shape dominated by finite complexes. Using tools of infinite-dimensional topology we can characterize SANR's rather simply. First we need a couple of definitions from [16] and a lemma.

A space X (always locally compact, separable and metric) is finitely dominated near ∞ if there is a finite complex K and a domination near ∞ , $f: K \times [0,1) \rightarrow X$, i.e., f is a proper map for which there is a compactum $A \subset X$ and a proper map $g: X - \text{int } A \rightarrow K \times [0,1)$ such that $f \circ g$ is proper homotopic to the inclusion $X - \text{int } A \hookrightarrow X$. If $f \circ g$ is only weakly proper homotopic to the inclusion $X - \text{int } A \hookrightarrow X$ we will say that X is weak finitely dominated near ∞ . Chapman and Ferry proved in

[16, Proposition 8.1] that if L is a locally finite CW complex which is weak finitely dominated near ∞ , then L is finitely dominated near ∞ . This will also be true for an arbitrary locally compact ANR, X , since R. Edwards [21] proved that $X \times Q$ is a Q -manifold and by Chapman's Triangulation Theorem [13], $X \times Q \cong K \times Q$ for some CW complex K .

Lemma 5.1. Let X be a finitely dominated near ∞ locally compact ANR. If X weak proper homotopy dominates ANR, Y , then Y is also finitely dominated near ∞ .

Proof. It suffices to see that Y is weak finitely dominated near ∞ . Let $f: K \times [0,1) \rightarrow X$, $A \subset X$, $g: X - \text{int } A \rightarrow K \times [0,1)$ and a proper homotopy $G: (X - \text{int } A) \times I \rightarrow X$ between $f \circ g$ and $i: X - \text{int } A \hookrightarrow X$ be given as in the above definition. In addition, let $\varphi: X \rightarrow Y$ and $\psi: Y \rightarrow X$ be proper maps such that $\varphi \circ \psi$ is weak proper homotopic to id_Y . Let $A' = \psi^{-1}(A)$ and define $f': K \times [0,1) \rightarrow Y$ to be $\varphi \circ f$ and $g': Y - \text{int } A' \rightarrow K \times [0,1)$ to be $g \circ \psi|_{Y - \text{int } A'}$. We must demonstrate that for any compactum $B' \subset Y$, there is a compactum $C' \subset Y - \text{int } A'$ and a homotopy H between $f' \circ g'$ and the inclusion $Y - \text{int } A' \hookrightarrow Y$ with $H((Y - \text{int } A') - C') \times I \subset Y - B'$. To find our C' , let $B = \varphi^{-1}(B')$, $D = \pi_X G^{-1}(B)$, where $\pi_X: X \times I \rightarrow X$ is a projection onto the first factor,

$D' = \psi^{-1}(D)$. Finally, there is a compactum C'' in Y and a homotopy $K: Y \times I \rightarrow Y$ between id_Y , and $\psi \circ \varphi$ such that $K((Y - C'') \times I) \subset Y - D'$. Then put $C' = (Y - \text{int } A') \cap C''$. A homotopy H is given by

$$H(y, t) = \begin{cases} K_{2t}(y) & , \quad 0 \leq t \leq 1/2 \\ \varphi \circ G_{2-2t} \circ \psi(y) & , \quad 1/2 \leq t \leq 1 \end{cases}$$

for every $y \in Y - \text{int } A'$. \square

Theorem 5.2. A Z -set X in a compact Q -manifold N is a SANR if and only if $M = N - X$ is finitely dominated near ∞ .

Proof. Using the trick from the proof of Corollary 4.3, embedding N in a copy of Q , we can effectively assume $N = Q$. Let $X \subset Q$ be a Z -set SANR . There is a Z -set and a finite complex $K \subset Q$ such that $N = Q - K$ weak proper homotopy dominates M . By Lemma 5.1, it suffices to prove that N is finitely dominated near ∞ . But this is certainly true because $K \times Q$ is a Z -set Q -manifold in the Hilbert cube $Q \times Q$ [45] so that there is an open collar around $K \times Q$ in $Q \times Q$ (see Lemma 2.6). Hence, $N \times Q \cong N$ [3] is finitely dominated near ∞ by K .

Conversely, suppose $M = Q - X$ is finitely dominated near ∞ . Then there is a finite complex K , a compactum $A \subset M$, proper maps $f: K \times [0, 1) \rightarrow M$, $g: M - \text{int } A \rightarrow K \times [0, 1)$,

and a proper homotopy $H: (M - \text{int } A) \times I \rightarrow M$ connecting the inclusion $M - \text{int } A \hookrightarrow M$ with $f \circ g$. Let $C_0(K)$ denote $K \times [0,1)$ with $K \times \{0\}$ identified to a point and let $\pi: K \times [0,1) \rightarrow C_0(K)$ be the projection map. A map $f': C_0(K) \rightarrow M$ is defined as follows: $f'(x,t) = f(x,t)$ if $1 > t \geq 1/2$, and $f'|_{K \times [0,1/2]/K \times \{0\}}$ is an extension of $f|_{K \times \{1/2\}}$ (recall, M is contractible). Observe that $f' \circ \pi$ and f are proper homotopic. We claim that $f' \circ g'$ is weak proper homotopic to id_M , where g' is an extension of $\pi \circ g: M - \text{int } A \rightarrow C_0(K)$ to a map $g': M \rightarrow C_0(K)$. Before we prove this, let us see how this implies that X is a SANR. The statement about $f' \circ g'$ assures that a contractible Q -manifold $C_0(K) \times Q$ weak proper homotopy dominates M . Since $(C_0(K) \times Q) \cup (K \times \{1\} \times Q) = \text{cone}(K) \times Q \cong Q$ [27] and $K \times \{1\} \times Q$ is a Z -set in $\text{cone}(K) \times Q$, Theorem 1.3 gives $\text{Sh}(K) = \text{Sh}(K \times Q) \geq \text{Sh}(X)$.

Thus, for an arbitrary compactum B in M , all we have to do is find a compactum C and a homotopy $G: M \times I \rightarrow M$ between id_M and $f' \circ g'$ that satisfies $G((M-C) \times I) \subset M-B$. Let $C' = A \cup \pi_{M-\text{int } A}^{-1}(B)$, where $\pi_{M-\text{int } A}: (M - \text{int } A) \times I \rightarrow M - \text{int } A$ is a projection onto the first factor. Let $F: [K \times [0,1)] \times [0,1] \rightarrow M$ be a proper homotopy between f and $f' \circ \pi$, and put $C'' = g^{-1}(\pi_{K \times [0,1)}^{-1}(B))$. The required G is any extension of the map $G': (M - \text{int } C) \times I \cup M \times \{0,1\} \rightarrow M$, where $C = C' \cup C''$,

defined below, to all of $M \times I$.

$$G'(x,t) = \begin{cases} x & \text{if } t = 0, \quad x \in M \\ f' \circ g'(x) & \text{if } t = 1, \quad x \in M \\ H_{2t}(x) & \text{if } 0 \leq t \leq 1/2, \quad x \in M - \text{int } C \\ F_{2t-1}(g(x)) & \text{if } 1/2 \leq t \leq 1, \quad x \in M - \text{int } C. \quad \square \end{cases}$$

Closely related to Theorem 5.2 is the following characterization of SANR's from [44].

Proposition 5.3. A 2-set X in a compact Q -manifold M is an SANR if and only if X admits I -regular open neighborhoods in M or, equivalently, if and only if there is a fundamental sequence $U_1 \supset U_2 \supset \dots$ of open neighborhoods of X in M and a finitely dominated complex K such that a diagram

$$\begin{array}{ccccccc} U_1 - X & \hookleftarrow & U_2 - X & \hookleftarrow & U_3 - X & \hookleftarrow & \dots \\ & \nwarrow & \nwarrow & \nwarrow & \nwarrow & & \\ & K & \xleftarrow{\text{id}} & K & \xleftarrow{\text{id}} & \dots \end{array}$$

homotopy commutes, i.e., M is docile at X .

Borsuk [10] defined a compactum $X \subseteq Q$ to be strongly movable if for every neighborhood U of X , there exists a neighborhood V of X such that for every neighborhood W of X , there is a homotopy $\varphi: V \times I \rightarrow U$ satisfying the following two conditions:

- (1) $\varphi_0(v) = v$, $\varphi_1(v) \in W$ for every $v \in V$,
 (2) $\varphi_1(x) = x$, for every point $x \in X$.

In the same paper, he proved the following.

Proposition 5.4. A compactum $X \subset Q$ is a SANR if and only if it is (Borsuk) strongly movable.

In the case X is a Z -set in Q we can prove Proposition 5.4 as follows. Assume X is a SANR. By Proposition 5.3, X admit I -regular open neighborhoods in Q . Therefore, the axiom $I - \text{Comp}(Q, X)$ from [43] holds. In other words, for every neighborhood U of X , there is $V \subset U$ such that for any neighborhood $W \subset U$, there is an isotopy $h_t: Q \rightarrow Q$ with $h_t|(Q-U) \cup N = \text{id}$, where N is a small neighborhood of X , and $h_1(V) \subset W$. Clearly, a homotopy $\varphi_t = h_t|_V$ satisfies (1) and (2) above. We shall use later the observation that a homotopy φ_t satisfies much stronger condition of keeping a neighborhood N of X pointwise fixed at all times.

Conversely, suppose a Z -set $X \subset Q$ is (Borsuk) strongly movable. We shall demonstrate that $Q-X$ is finitely dominated near ∞ .

Let $Q = V_0 \supset V = V_1 \supset V_2 \supset \dots$ be a decreasing sequence of prism neighborhoods of X in Q and let, for every $i > 0$, $\varphi_t^i: V_i \times I \rightarrow V_{i-1}$ be a homotopy for which $\varphi_0^i = \text{id}$, $\varphi_1^i(V_i) \subset V_{i+1}$, and $\varphi_1^i|_X = \text{id}$. We can select

neighborhoods V_i such that there is a homotopy $\lambda_t: Q \rightarrow Q$ with the properties: $\lambda_0 = \text{id}$, $\lambda_t(Q) \subset Q-X$ for every $t > 0$, and $\lambda_t(V_{i-1}) \subset V_{i-1}$, $\lambda_t(Q-V_{i-2}) \subset Q-V_{i-1}$ whenever $t \leq 1/i$. If $r: V \rightarrow [0, \infty]$ is a continuous map satisfying $r^{-1}(\infty) = X$, define a proper map $g: V-X \rightarrow V \times [0, \infty)$ by $f(v) = (v, r(v))$. Next, construct a proper map $f: V \times [0, \infty) \rightarrow Q-X$ setting $f(v, t) = \lambda_{1/n} \circ \varphi_1^1 \circ \dots \circ \varphi_1^n \circ \varphi_{t-n}^{n+1}(v)$, whenever $n \leq t \leq n+1$. Then it is easy to check that $f \circ g$ is proper homotopic to the inclusion $V-X \hookrightarrow Q-X$.

Note. In the above definition we can, in addition, assume $\varphi_1|_N = \text{id}$ for some neighborhood $N \subset W$ of X . Indeed, on the closed subset $T = V \times \{0\} \cup \tilde{W} \times \{1\} \cup X \times [0, 1]$, of $V \times [0, 1]$ (where \tilde{W} is a closed neighborhood of X contained in W) we can define a map φ into W by

$$\varphi(v, t) = \begin{cases} \varphi_1(v) & , \quad t = 0 \\ v & , \quad t = 1 \text{ or } v \in X . \end{cases}$$

and because W can always be assumed an ANR, there is a neighborhood \tilde{N} of T in $V \times [0, 1]$ and an extension $\phi: \tilde{N} \rightarrow W$ of φ . Let $N_1 \supset \text{cl } N \supset N$ be two open neighborhoods of X with $N_1 \times I \subset \tilde{N}$ and let $r: V \rightarrow [0, 1]$ be a continuous function such that $r|_{V-N_1} = 0$ and $r|_N = 1$. Now, define a homotopy $H: V \times [0, 1] \rightarrow W$ as

$$H(v, t) = \phi(v, t \cdot r(v)) .$$

Then, $H_0 = \varphi_1$ and $H_1|_N = \text{id}$.

With this simple observation, using techniques of Chapters III and IV, we can establish the following.

Theorem 5.5. A Z-set $X \subseteq Q$ is a SANR if and only if the complement $M = Q - X$ has the following property:

(NR) For every compactum $A \subseteq M$ there is a compactum $D \supset C$ so that given any map $\varphi: (K, L) \rightarrow (M - B, M - D)$ of a pair (K, L) of finite polyhedra there is a homotopy $H: K \times I \rightarrow M - A$ with $H_0 = \varphi$, $H_1(K) \subseteq M - C$, and $H_1|_L = \varphi|_L$.

Definition 5.6. A locally compact metric space M is said to belong to a class $\text{SANR}(\infty)$ if M is an $\text{ANR}(\mathcal{M})$ and has property (NR) . By restricting a dimension of K in (NR) to be $\leq n$ we will get a class $\text{SANR}_n(\infty)$ of locally compact metric ANR's .

Lemma 5.7. If an $\text{ANR}(\mathcal{M})$ M is homotopy dominated at ∞ by an $\text{SANR}(\infty)$ N , then M is also an $\text{SANR}(\infty)$.

Proof. Let $f: N \rightarrow M$ and $g: M \rightarrow N$ be proper maps such that for every compactum X in M , there is a compactum $X_0 \supset X$ in M and a homotopy $H_X: (M - X_0) \times I \rightarrow M - X$ between $\text{id}_{M - X_0}$ and $f \circ g|_{M - X_0}$. Given a compactum $A \subseteq M$, $f^{-1}(A_0) = A'$ is a compactum in N . Pick $B' \supset A'$ using the fact that

$N \in \text{SANR}(\infty)$, and put $B = A \cup g^{-1}(B')$. If $C \supset B$ is an arbitrary compactum, $C' = B' \cup f^{-1}(C_0)$ is a compactum in N so that there is $D' \supset C'$ satisfying (NR) . Let D be the union of C_0 and $g^{-1}(D')$.

Assume $\varphi: (K, L) \rightarrow (M-B, M-D)$ is a map of a pair (K, L) of finite polyhedra. By assumption, there is a homotopy $\phi_t: K \rightarrow N-A'$ with $\phi_0 = g \circ \varphi$, $\phi_1(K) \subset N-C'$ and $\phi_1|_L = g \circ \varphi|_L$. Define $\psi_t: K \rightarrow M-A_0$ as $f \circ \phi_t$. Then $\psi_0 = f \circ g \circ \varphi$, $\psi_1(K) \subset M-C_0$ and $\psi_1|_L = f \circ g \circ \varphi|_L$. A homotopy $H_A \circ \varphi$ connects φ with $f \circ g \circ \varphi$ and $H_C \circ (\varphi|_L)$ joins $\varphi|_L$ with $f \circ g \circ (\varphi|_L)$. Since $M-A$ is an $\text{ANR}(\mathcal{M})$ we can apply the Homotopy Extension Theorem to arrive to a required homotopy. \square

We will say that a compactum X is a SANR_n -space provided a complement of a Z -set copy of X in Q is in class $\text{SANR}_n(\infty)$. The proof of the previous lemma shows that this is a shape invariant property.

Theorem 5.8. If X is an LC^{n-1} compactum and a Z -set in Q , then $M = Q-X \in \text{SANR}_n(\infty)$.

Proof. We will use the full strength of [28] already applied in the proof of Proposition 4.10. Let $A \subset M$ be an arbitrary compactum. Select $B \supset A$ as in Proposition 4.10. If C is a compactum in M containing B , we can find $D \supset C$ in the analogous way to how B was chosen with respect to A .

Suppose $\varphi: (K, L) \rightarrow (M-B, M-D)$ is a map of a pair (K, L) of finite polyhedra, $\dim K \leq n$, into $(M-B, M-D)$. A restriction $\varphi|_L$ is homotopic in $M-C$, to a map $\tilde{\varphi}: L \rightarrow X$ by the choice of D . Hence, by a homotopy extension theorem, φ is homotopic in $(Q-B, Q-C)$ to a map $\psi: (K, L) \rightarrow (Q-B, X)$. Then ψ is homotopic in $Q-A$ to a map $\psi': K \rightarrow Q-C$, where $\psi'|_L = \psi|_L$. As in Proposition 4.10, using small homotopies that slide Q off of X we will get a required homotopy. \square

Lemma 5.9. If X is a Z -set SANR in Q , then Q/X is an ANR.

Proof. This follows immediately from [26] with the use of Proposition 5.3. \square

Theorem 5.10. Let X be an SANR and $A \subset X$ a closed subset and an SANR. Then X/A is also an SANR.

Proof. Consider X as a Z -set in Q . Combining Lemma 5.9 and a recent Edwards' theorem [21], $X/A \times \{0\}$ is a Z -set copy of X/A in a Q -manifold $Q/A \times Q$. Since Q is docile at X (see Proposition 5.3), it is easy to verify that $Q/A \times Q$ is docile at $X/A \times \{0\}$, so that, by Proposition 5.3, X/A is an SANR. \square

The following two corollaries give partial answers to Borsuk's question [11]: "Is it true that $X, Y, X \cap Y \in \text{SANR}$

implies $X \cup Y \in \text{SANR}$ and its converse?

Corollary 5.11. Let Z be the union of a SANR , X_1 , and a set of trivial shape X_2 such that $X_0 = X_1 \cap X_2$ is an SANR . Then Z is an SANR .

Proof. It is well known that $\text{Sh}(Z) = \text{Sh}(Z/X_2)$. But $Z/X_2 \cong X_1/X_0$, and we can apply Theorem 5.10. \square

Corollary 5.12. Let Z be the union of compacta X_1 and X_2 such that $X_0 = X_1 \cap X_2$ has trivial shape. If Z is an SANR , then both X_1 and X_2 are SANR 's .

Proof. Again, $\text{Sh}(Z) = \text{Sh}(Z/X_0)$ and X_1/X_0 is a retract of Z/X_0 (simply collapse all of X_2/X_0 to a point \hat{X}_0) . Hence, X_1/X_0 is an SANR . Therefore, X_1 is an SANR because $\text{Sh}(X_1) = \text{Sh}(X_1/X_0)$. \square

Now, we shall describe a method of establishing when a locally finite CW complex L is n -dominated near ∞ , i.e., when we can find an n -dimensional finite complex M and a domination near ∞ , $d: M \times [0,1) \rightarrow L$ (Lemma 5.13) and then apply this to get a characterization of shape dimension of SANR 's (Theorem 5.14).

Let $f: K \times [0,1) \rightarrow L_1 \hookrightarrow L$, $g: L_1 \rightarrow K \times [0,1)$, and $\alpha_u: K \rightarrow K$ be as in the Definition 4.4 of [16] (see also diagram (**) below).

Lemma 5.13. L is n -dominated near ∞ if and only if

there is an n -dimensional finite complex M such that the diagram

$$(*) \quad \begin{array}{ccc} K & \xrightarrow{\alpha_u} & K \\ & \searrow \varphi & \nearrow \psi \\ & M & \end{array}$$

homotopy commutes.

Proof. Suppose that there are maps φ and ψ making the diagram $(*)$ homotopy commutative. Let $L_2 \subset L_1$ be a subcomplex for which $f \circ g|_{L_2}$ is homotopic in L_1 to the inclusion $L_2 \hookrightarrow L_1$ and let $u \in [0,1)$ be such that $f(K \times \{u\}) \subset L_2$. Put $a = f \circ (xu) \circ \psi \circ p_M : M \times [0,1) \rightarrow L_2$ and $b = (xv) \circ \varphi \circ p_K \circ (g|_{L_2}) : L_2 \rightarrow M \times [0,1)$.

$$(**) \quad \begin{array}{ccccccc} & & & \alpha_u & & & \\ & & & \curvearrowright & & & \\ & & L_2 & \searrow g|_{L_2} & & & \\ & & \downarrow & & & & \\ K & \xrightarrow{xu} & K \times [0,1) & \xrightarrow{f} & L_1 & \xrightarrow{g} & K \times [0,1) \xrightarrow{p_K} K \\ & \searrow \varphi & & & & & \nearrow \psi \\ & & M & & & & \\ & & \downarrow xv \quad \uparrow p_M & & & & \\ & & M \times [0,1) & & & & \end{array}$$

We claim that $a \circ b$ is properly homotopic to the inclusion $L_2 \hookrightarrow L$. Note that $a \circ b = f \circ (\alpha_u \times c_u) \circ (g|_{L_2}) = \gamma$,

where $c_u: [0,1) \rightarrow [0,1)$ is a constant map into u . Hence, the value $\gamma(l)$, for any $l \in L_2$, is obtained as follows:
 $l \mapsto (p_K g(l), p_{[0,1)} g(l)) \mapsto ((p_K g f(p_K g(l), u), u), u) \mapsto$
 $f((p_K g f(p_K g(l), u), u), u)$. Using homotopies only on $[0,1)$ direction we can easily see that γ is proper homotopic in L to a composition $f \circ g \circ f \circ (g|_{L_2})$. But this later can be written as $(f \circ g) \circ (f \circ (g|_{L_2}))$ and by assumption is proper homotopic in L to the inclusion $L_2 \hookrightarrow L$.

Conversely, if L is n -dominated near ∞ by M^n and $f: K \times [0,1) \rightarrow L_1 \hookrightarrow L$ is an arbitrary domination near ∞ with the associated map $\alpha_u: K \rightarrow K$, we claim that α_u factors through M . Indeed, without loss of generality, we can assume that there are proper maps $f_1: M \times [0,1) \rightarrow L_1$ and $g_1: L_1 \rightarrow M \times [0,1)$ such that $f_1 \circ g_1$ is proper homotopic to the inclusion $L_1 \hookrightarrow L$. Then $\varphi = p_M \circ g_1 \circ f \circ (xu): K \rightarrow M$ and $\psi = p_K \circ g \circ f_1 \circ (xv): M \rightarrow K$ satisfy $\psi \circ \varphi = \alpha_u$. \square

Theorem 5.14. A Z -set and an SANR, X , in Q has a shape dimension $\leq n$ if and only if $M = Q - X$ is dominated near ∞ by an n -dimensional finite complex.

Proof. If M is n -dominated near ∞ , then it is easy to verify that M is n -tame at ∞ , and therefore that $Sd(X) \leq n$.

On the other hand, if a Z -set and an SANR, X , in Q has a shape dimension $\leq n$, we will prove that the map

α_u associated to, a domination near ∞ , $f:K \times [0,1) \rightarrow M$ as defined above can be factored through an n -dimensional finite complex. By Lemma 5.13, it would follow that M is n -dominated near ∞ . Let $A \subset M$ be a compactum for which there is a proper map $g:M - \text{int } A \rightarrow K \times [0,1)$ with $f \circ g$ proper homotopic to the inclusion $M - \text{int } A \hookrightarrow M$. Since M is n -tame at ∞ , we can find an n -complex L and a compactum $B \supset A$ in M such that $M - \text{int } B \hookrightarrow M - \text{int } A$ factors through L . Pick a real number $u \in [0,1)$ for which $f(K \times \{u\}) \subset M - \text{int } B$. Then $\varphi = \alpha \circ f \circ (xu):K \rightarrow L$ and $\psi = P_K \circ g \circ \beta:L \rightarrow K$ give a factorization of α_u through L , where u and β make

$$\begin{array}{ccc}
 M - \text{int } B & \hookrightarrow & M - \text{int } A \\
 \searrow \alpha & & \nearrow \beta \\
 & L &
 \end{array}$$

homotopy commutative. \square

Corollary 5.15. Let L be a locally finite CW complex and let L be finitely dominated near ∞ . Then L is n -tame at ∞ if and only if L is n -dominated near ∞ .

Proof. Assume L is dominated near ∞ by an n -dimensional finite complex K . Let $f:K \times [0,1) \rightarrow L_1 \hookrightarrow L$ and $g:L_1 \rightarrow K \times [0,1)$ be as above. Then, with the notation from

the proof of Lemma 5.13,

$$\begin{array}{ccc}
 L_1 & \hookrightarrow & L \\
 p_K \circ g \searrow & & \nearrow f \circ (xu) \\
 & K &
 \end{array}$$

is a factorization of $L_1 \hookrightarrow L$ through K . It is clear that this can be accomplished in any neighborhood of ∞ . Hence, L is n -tame at ∞ .

Conversely, suppose L is dominated near ∞ by a finite complex K and also that L is n -tame at ∞ . We shall show that α_u factors through an n -dimensional finite complex. Lemma 5.13 implies, then, L is n -dominated near ∞ .

In the diagram (**), select L_2 in such a way that

$$\begin{array}{ccc}
 L_2 & \hookrightarrow & L_1 \\
 \alpha \searrow & & \nearrow \beta \\
 & P &
 \end{array}$$

homotopy commutes for some n -dimensional finite complex P .

Let $u \in (0,1)$ be so close to 1 that $f(K \times \{u\}) \subset L_2$.

Then

$$\begin{array}{ccc}
 K & \xrightarrow{\alpha_u} & K \\
 \alpha \circ f \circ (xu) \searrow & & \nearrow p_K \circ g \circ \beta \\
 & P &
 \end{array}$$

is a required factorization of α_u . \square

CHAPTER VI

COMPACTA OF TRIVIAL SHAPE

In this chapter, we will consider locally compact non-compact spaces trivial at ∞ and compacta with the shape of a point, i.e., shape absolute retracts (SAR's) .

Lemma 6.1. A Z-set $X \subset Q$ is an SAR if and only if the complement $M = Q - X$ is trivial at ∞ , i.e., for any compactum $A \subset M$, there is a compactum $B \supset A$ such that every map $\varphi: S^k \rightarrow M - B$ of a k -sphere S^k ($k \geq 0$) is null-homotopic in $M - A$.

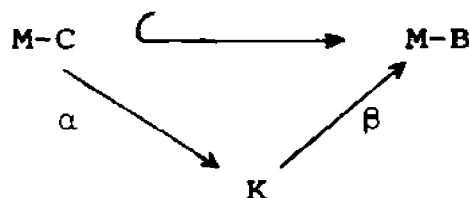
Proof. If X is an SAR , $Q - X \cong Q \times [0,1)$ [12] is clearly trivial at ∞ . Conversely, if M is trivial at ∞ , then, by a technique of Chapters III - V , we can see that X is contractible inside each of its neighborhoods, and thus is an SAR [6, 26]. \square

Restricting the dimension k in Lemma 6.1 to be $\leq n$, we get the notion of an n -trivial at ∞ locally compact metric space. This concept can be used for characterizing UV^n -compacta [46], or, equivalently, approximatively

n -connected compacta [10].

Theorem 6.2. If M is n -tame at ∞ and n -trivial at ∞ , then M is trivial at ∞ .

Proof. Take any compactum A in M and pick $B \supset A$ using n -triviality of M at ∞ . Let $C \supset B$ such that



homotopy commutes for some n -dimensional finite complex K . Given a map $\varphi: P \rightarrow M-C$, we see that φ is null-homotopic in $M-A$ since β is and $\varphi \simeq \beta \circ \alpha \circ \varphi$. \square

Theorem 6.3. If M is movable and n -trivial at ∞ , for every $n > 0$, then M is trivial at ∞ .

Proof. Let A be a compactum in M . Pick $B \supset A$ so that the condition (M) of Chapter IV holds. One can easily check that a compactum B suffices since every map into $M-B$ can be homotoped outside A to a map whose image is arbitrarily close to ∞ . \square

Corollary 6.4 ([10]). A compactum X has trivial shape if and only if X is movable and approximatively n -connected for all $n = 0, 1, 2, \dots$.

Theorem 6.5. A Z -set X in Q has trivial shape if and only if

- (a) $Sd(X) < \infty$,
- (b) X is approximatively 1-connected, i.e., the complement $M = Q - X$ is 1-trivial at ∞ , and
- (c) the reduced Čech cohomology $\check{H}^*(X)$ of X is zero.

Proof. It is clear that every Z -set of trivial shape in Q has properties (a) - (c). Conversely, by (b), X is connected so that M has exactly one end. Observe that M is contractible and in light of (a), proper homotopy equivalent to a locally finite polyhedron of finite dimension. Let $\underline{f}: X \rightarrow \{\text{pt}\}$ be a constant fundamental sequence. We will prove that \underline{f} is a fundamental equivalence. It follows from Theorem 1.3 that \underline{f} determines a proper map $f: M \rightarrow Q - \{\text{pt}\}$. But f induces isomorphisms $H_*(M) \rightarrow H_*(Q - \{\text{pt}\})$, $H_C^*(M) \rightarrow H_C^*(Q - \{\text{pt}\})$, where H_* denotes a singular homology and H_C^* a singular cohomology with compact supports, since $H_C^*(M)$ is canonically isomorphic to $\bar{H}_C^*(M)$, the Alexander cohomology with compact support ([47, p. 341]), and $\bar{H}_C^*(M) \cong \bar{H}^*(Q, X) \cong \check{H}^*(Q, X) \cong \check{H}^*(X)$ ([47, p. 321]). Now, we can apply Corollary 4.9 in [22] to conclude that f is a proper homotopy equivalence and, therefore that \underline{f} is a fundamental equivalence by Theorem 1.3 again. \square

Corollary 6.6. Let $X \subset Q$ be an approximatively 1-connected Z -set of a finite shape dimension. If Q/X has zero reduced Čech cohomology, then X has trivial shape.

Proof. This follows immediately from the isomorphisms $\bar{H}^*(Q/X) \cong \tilde{H}^*(Q/X)$ and $\tilde{H}^*(Q/X) \cong \bar{H}_C^*(Q-X)$ [47] and Theorem 6.5. \square

Proposition 6.7 ([48]). Let X be a compactum with $\text{Sh}(X) < \text{Sh}(S^n)$. Then X has a trivial shape.

Proof. Consider S^n and X as Z -sets in Q . Observe that $\bar{H}_C^*(Q-S^n) \cong \bar{H}^*(Q, S^n) \cong \bar{H}^*(S^n)$. If $\text{Sh}(X) < \text{Sh}(S^n)$, then there is a fundamental sequence $f: S^n \rightarrow X$ that is not a fundamental equivalence, but for which there is $g: X \rightarrow S^n$ such that $f \circ g \simeq \text{id}_X$. By Theorem 1.3, we can find proper maps $f: Q-S^n \rightarrow Q-X$ and $g: Q-X \rightarrow Q-S^n$ such that f is not a weak proper homotopy equivalence but $f \circ g$ is weakly proper homotopic to id_{Q-X} . Note that $Q-S^n$ is finitely dominated near ∞ . Hence $Q-X$ is also finitely dominated near ∞ . Therefore, we can assume that $f \circ g$ is properly homotopic to id_{Q-X} [16, Section 3]. This means that $g^* \circ f^* = (f \circ g)^*: \bar{H}_C^*(Q-X) \rightarrow \bar{H}_C^*(Q-X)$ is the identity, i.e., $\bar{H}_C^*(Q-X)$ is a direct summand of

$$\bar{H}_C^*(S^n) = \begin{cases} \mathbb{Z}, & q = n, 0 \\ 0, & q \neq n, 0. \end{cases}$$

If $\bar{H}_C^n(Q-X) = 0$, then we are done by Theorem 6.5. The other possibility is $\bar{H}_C^n(Q-X) = \mathbb{Z}$. We will prove that this is impossible. First, note that, by Corollary 4.9 in [22], neither f_n^* nor g_n^* can be isomorphisms. Let $g_n^*(1) = \theta$ and $f_n^*(1) = \eta$ where 1 denotes unit in $\bar{H}^n(Q-S^n)$ and $\bar{H}^n(Q-X)$, respectively. Note $|\theta| > 1$ since g_n^* is not an isomorphism. Then g_n^* cannot be onto as a relation $g_n^* \circ f_n^* = \text{id}$ requires. \square

CHAPTER VII

UVW^n -SPACES

Armentrout [49], introduced property UVW^n of compact spaces in order to extend classical theorems about decompositions of ANR's . He asked if this property is a shape invariant and how it relates to SANR's . In this chapter, we will answer two of his questions affirmatively (see Corollary 7.2 and Theorem 7.3).

If $n \geq 0$ is an integer, a closed subset X of Q has property UVW^n if and only if, for each open neighborhood U of X , there is an open neighborhood V of X such that, for each open neighborhood W of X , there is an open neighborhood Z of X such that, for each k , with $0 \leq k \leq n$ and each map $f: (B^k, Bd B^k) \rightarrow (V, Z)$, there is a homotopy $H: (B^k \times I, (Bd B^k) \times I) \rightarrow (U, W)$ connecting f with a map H_1 for which $(H_1(B^k), H_1(Bd B^k)) \subset (W, Z)$. Let UVW^∞ mean UVW^n for all $n \geq 0$.

Lemma 7.1. A Z-set X in Q has property UVW^n if and only if the complement $M = Q - X$ satisfies:

(A) for each compactum A in M , there is a

compactum $B \supset A$ such that, for each compactum $C \supset A$ in M , we can find a compactum $D \supset C$ such that, for each k with $0 \leq k \leq n$, and each map $f: (B^k, Bd B^k) \rightarrow (M-B, M-D)$, there is a homotopy $H: (B^k \times I, (Bd B^k) \times I) \rightarrow (M-A, M-C)$ of f such that $(H_1(B^k), H_1(Bd B^k)) \subset (M-C, M-D)$.

Proof. Let X be a Z -set in Q and assume X has property UVW^n . If A and C , $A \subset C$, are arbitrary compacta in $M = Q - X$, $U = Q - A$ and $W = Q - C$ are open neighborhoods of X in Q . Pick open neighborhoods V and Z as in the above definition and put $B = Q - V$ and $D = Q - Z$. Compacta A, B, C, D will satisfy (A) as is easily verified using Chapman's trick [12] of deforming Q off of X moving points only inside Z .

The converse is equally routine and therefore left to the reader. \square

It is clear that if a space M has property (A) and $h: M \rightarrow N$ is a homeomorphism, then N also has property (A). Hence, since two shape equivalent Z -sets in Q have homeomorphic complements (Theorem 1.4), we immediately get

Corollary 7.2. The property UVW^n of compacta is a shape invariant. \square

Theorem 7.3. Every SANR, X , has property UVW^∞ .

Proof. Consider X as a Z -set in Q . By Theorem 5.2,

the complement $M = Q - X$ is finitely dominated near ∞ .
 Let $f: K \times [0, 1) \rightarrow M$ be a domination near ∞ , and select a compactum $A' \subset M$ for which there is a proper map $g: M - \text{int } A' \rightarrow K \times [0, 1)$ such that $f \circ g$ is properly homotopic to the inclusion $M - \text{int } A' \hookrightarrow M$ via a proper homotopy h_t . If $A \subset C$ are arbitrary compacta in M , we can find $B \supset A \cup A'$ and $D \supset C \cup A'$ with $h_t(M - D) \subset M - C$ and $h_t(M - B) \subset M - A$. Now, we will show that $f \circ g|_{M - B}$ is homotopic to a map $(M - B, M - D) \rightarrow (M - C, M - D)$ by a homotopy in $(M - A, M - C)$. To see this, pick numbers $t_A < t_B$ and $t_C < t_D$ with $t_C > t_A$ and $t_D > t_B$ in $[0, 1)$ such that $f^{-1}(Y) \subset K \times [0, t_Y]$ for each $Y \in \{A, B, C, D\}$. Let $\lambda_t: [0, 1) \rightarrow [0, 1)$ be a homotopy such that $\lambda_0 = \text{id}$, $\lambda_1([t_B, 1)) \subset [t_C, 1)$, $\lambda_1([t_C, 1)) \subset [t_D, 1)$, $\lambda_t|_{[0, t_A]} = \text{id}$, $\lambda_t([t_C, 1)) \subset [t_C, 1)$. Then $f \circ \lambda_t \circ g|_{M - B}$ is the required homotopy. Hence, we have proved that, for every compactum A in M , there is a compactum $B \supset A$ such that, for any compactum $C \supset A$, we can find a compactum $D \supset C \cup B$ and a homotopy $h_t: M - B \rightarrow M - A$ with h_0 - the inclusion $M - B \hookrightarrow M - A$, $h_t(M - D) \subset M - C$, and $(h_1(M - B), h_1(M - D)) \subset (M - C, M - D)$. It is clear that this implies that M has property (A). \square

Theorem 7.4. If a compactum X is approximatively k -connected for all $0 \leq k \leq n$, then X is a UVW^n -space.

Proof. Considering X as a Z -set in Q , we will prove that $M = Q - X$ has the property (Q). Let A be a compactum in M . Pick $B \supset A$ such that every map $S^k \rightarrow M - B$, $0 \leq k \leq n$, is null-homotopic in $M - A$. If $C \supset A$ is a compactum, select $D \supset C$ so that every map $S^k \rightarrow M - D$, $0 \leq k \leq n$, is null-homotopic in $M - (B \cup C)$.

Now, let $f: (B^k, \text{Bd } B^k) \rightarrow (M - B, M - D)$, $0 \leq k \leq n$, be given. By assumption, there is a homotopy $h_t: \text{Bd } B^k \rightarrow M - C$ with $h_0 = f|_{\text{Bd } B^k}$ and $h_1(\text{Bd } B^k) = p = \text{point} \in M - D$ (observe that $h_t(\text{Bd } B^k)$ remains in the same path-component K of $M - C$ and $K \cup (\text{path-component of } M - D \text{ containing } h_0(\text{Bd } B^k))$ is path-connected). Define a map φ on the boundary of $B^k \times I$ by

$$\varphi(x, t) = \begin{cases} p & , \quad t = 1 \\ h_t(x) & , \quad x \in \text{Bd } B^k, \quad t \in I \\ f(x) & , \quad t = 0 . \end{cases}$$

Again, φ extends to a map $\tilde{\varphi}: B^k \times I \rightarrow M - A$ that gives a required homotopy of f . \square

We leave the question if every compactum X which has property UVW^∞ is an SANR unanswered and only make some comments on how that question is related to the following (converse of Lemma 5.9).

Conjecture 7.5. Let A be a Z -set in Q such that Q/A is an ANR. Then A is an SANR.

Let X be an n -dimensional ($n < \infty$) compactum with property $UVW^{(2n+1)}$. Consider X as a subset of a $(2n+1)$ -dimensional closed cell B^{2n+1} . It follows from Theorem 2 in [49] that B^{2n+1}/X is an ANR. Applying the main result of [26], we see that Q/X' is an ANR, where X' is a copy of X in Q . Therefore, if Conjecture 7.5 is a true statement, then at least all finite-dimensional UVW^∞ -compacta are SANR's. On the other hand, the existence of a finite-dimensional UVW^∞ -compactum that is not an SANR gives a counterexample for Conjecture 7.5.

CHAPTER VIII

HOMOLOGY AND HOMOTOPY PRO-GROUPS

In this chapter, we shall briefly describe a method of introducing algebraic techniques in our study of homotopy properties of non-compact locally compact metric spaces. In a natural way, we shall define certain inverse systems of groups associated with every locally compact non-compact metric space. The language of pro-categories will be used to accomplish this. We shall assume that the reader is familiar with results and terminology employed in this chapter (see [38] and [20] for references).

Let M be a locally compact metric space. We can think about M as an element \underline{M} of $\text{pro-}\mathcal{X}$, where \mathcal{X} is a category of topological spaces and homotopy classes of continuous maps. Indeed, let \mathcal{C}_M be the set of all compact subsets of M directed by inclusions. Then $\underline{M} = \{M_A = M - A, [i_{B,A}]; \mathcal{C}_M\}$ is an inverse system with bonding maps homotopy classes of inclusions $i_{B,A}: M - B \hookrightarrow M - A$. Every proper map $f: M \rightarrow N$ induces a morphism $\underline{f}: \underline{M} \rightarrow \underline{N}$ in $\text{pro-}\mathcal{X}$. The index function of \underline{f} is a map $\varphi: \mathcal{C}_N \rightarrow \mathcal{C}_M$

given by $\varphi(B) = f^{-1}(B)$, and coordinate maps $f_B: M_{\varphi(B)} \rightarrow N_B$ are simply homotopy classes of restrictions of f from $M-f^{-1}(B)$ into $N-B$.

Lemma 8.1. Proper maps $f, g: M \rightarrow N$ are homotopic at ∞ iff $\underline{f}, \underline{g}$ are homotopic maps of systems.

Proof. Assume $f \sim g$. Given a compactum B in N , we must find a compactum $A \supset f^{-1}(B) \cup g^{-1}(B)$ in M such that

$$f \circ i_{A, f^{-1}(B)} \simeq g \circ j_{A, g^{-1}(B)}$$

in $N-B$. A is just a compactum containing $f^{-1}(B)$ and $g^{-1}(B)$ so that $f|_{M-A}$ is homotopic in $N-B$ to $g|_{M-A}$.

Conversely, if $\underline{f} \simeq \underline{g}$, then clearly for any compactum $B \subset N$, we can find $A \subset M$ so that $f|_{M-A}$ is homotopic in $N-B$ to $g|_{M-A}$. \square

Let G be an abelian group, and $k \geq 0$ an integer. Then

$$H_k(M; G) = \{H_k(M_A; G), (i_{B,A})_{k*}, \mathcal{C}_M\}$$

is an inverse system of groups, i.e., an object of the category $\text{pro-}\mathcal{G}$ of pro-groups called the k -th homology pro-group of M with coefficient in G . A proper map $f: M \rightarrow N$ induces a morphism of pro-groups $f_{k*}: H_k(M; G) \rightarrow H_k(N; G)$ in a natural way. It is readily seen that $(gf)_{k*} = g_{k*} \circ f_{k*}$ and

$\text{id}_{k*} = \text{id}$. Therefore, two spaces that are homotopy equivalent at ∞ have isomorphic homology pro-groups.

In order to define analogues of homotopy groups, we shall consider only one-ended locally compact metric spaces M . Then \mathcal{C}_M has a cofinal subset \mathcal{C}_M^* consisting of all compacta $A \subset M$ for which $M-A$ is connected. For every $A \in \mathcal{C}_M^*$, pick a point $x_A \in M-A$ and if B and A , $B \supset A$, are both in \mathcal{C}_M^* , connect points x_A , x_B by a path $\alpha_{B,A}: I \rightarrow M-A$. Now, define an inverse system

$$\pi_k(M, \{x_A\}, \{\alpha_{B,A}\}) = \{\pi_k(M_A, x_A), (\alpha_{B,A})_* \circ (i_{B,A})_{k*}, \mathcal{C}_M^*\}$$

of groups, where $(\alpha_{B,A})_k$ is an isomorphism $\pi_k(M_A, x_B) \rightarrow \pi_k(M_A, x_A)$ induced by a path $\alpha_{B,A}$ [50, p. 126]. Of course, systems $\pi_k(M, \{x_A\}, \{\alpha_{B,A}\})$ depend on the choice of points $\{x_A\}$ and paths $\{\alpha_{B,A}\}$. In case M is 1-trivial at ∞ , all such choices lead to systems that have the same homotopy type [36].

Suppose now that M is an ANR . If B and A , $B \supset A$, are compacta in \mathcal{C}_M^* , by the Homotopy Extension Theorem, there is a map $P_{B,A}: M_B \rightarrow M_A$ homotopic to $i_{B,A}$ and such that $P_{B,A}(x_B) = x_A$. Hence, the inverse system $\mu = \{M_A, [P_{B,A}], \mathcal{C}_M^*\}$ is equivalent to \underline{M} and bonding maps of μ map base points into base points. Then we can define k -th homotopy pro-group of M to be the k -th homotopy pro-group of the inverse system μ as defined in [38], i.e.,

$$\pi_k(M, \{x_A\}) = \{\pi_k(M_A, x_A), (p_{B,A})_{k*}, C_M^*\}.$$

For a proper map $f: M \rightarrow N$ between one-ended ANR's, we can select base points $\{y_B\}$ in N and $\{x_A\}$ in M so that f induces in a natural way a homomorphism $f_{k*}: \pi_k(M, \{x_A\}) \rightarrow \pi_k(N, \{y_B\})$; in order to accomplish this we might need to pass to a cofinal subset of C_M^* .

Theorem 3.2. A one-ended ANR, M , is n -trivial at ∞ iff $\pi_k(M, \{x_A\}) = 0$ for some set of base points $\{x_A\}$ and all $1 \leq k \leq n$.

Proof. Let $\pi_k(M, \{x_A\}) = 0$, $1 \leq k \leq n$, and let $A \in C_M^*$. Pick $B \supset A$ in C_M^* so that $(p_{B,A})_{k*}$ is a zero map, for all $1 \leq k \leq n$. Take any map $\varphi: S^k \rightarrow M_B$. By the homotopy extension theorem, φ is homotopic in M_B to a map $\psi: (S^k, s_0) \rightarrow (M_B, x_B)$, where s_0 is a north pole of S^k . But,

$$\text{constant map} \simeq p_{B,A} \circ \psi \simeq i_{B,A} \circ \psi = \psi \simeq \varphi,$$

which shows that M is n -trivial at ∞ .

Conversely, assume M is n -trivial at ∞ and take any compactum $A \in C_M^*$. Let $B \supset B_1 \supset A$; $B, B_1 \in C_M^*$, be such that every map $\varphi: S^k \rightarrow M_B$ ($1 \leq k \leq n$) is null-homotopic in M_{B_1} and every loop in M_{B_1} is trivial in M_A . We shall prove that, for a pointed map $\psi: (S^k, s_0) \rightarrow (M_B, x_B)$,

$\chi = p_{B,B_1} \circ \psi$ is homotopic in M_A to a constant map into x_{B_1} by a homotopy that maps s_0 into x_{B_1} at all times.

Let $H: S^k \times I \rightarrow M_{B_1}$ be a null-homotopy of χ . The existence of H follows from $\chi = p_{B,B_1} \circ \psi \simeq i_{B,B_1} \circ \psi = \psi$. Let $\alpha: I \rightarrow M_{B_1}$ be a path connecting points $H_1(s_0)$ and $\chi(s_0)$. On the boundary of $\{s_0\} \times I \times I$ define a map g into M_{B_1} by

$$g(s_0, a, b) = \begin{cases} \chi(s_0) & , \quad b = 1 \quad \text{or} \quad a = 0 \\ H_a(s_0) & , \quad b = 0 \\ \alpha(b) & , \quad a = 1 \end{cases} .$$

The choice of B gives us an extension $G: \{s_0\} \times I \times I \rightarrow M_B$ of g . Define a partial homotopy on $(S^k \times I) \times \{0\} \cup S^k \times \{0\} \times I \cup S^k \times \{1\} \times I \cup \{s_0\} \times I \times I$ by H on $(S^k \times I) \times \{0\}$, by G on $\{s_0\} \times I \times I$, by χ level-wise on $S^k \times \{0\} \times I$, and by α level-wise on $S^k \times \{1\} \times I$. Applying the homotopy extension theorem, we will get a map $S^k \times I \times I$ whose restriction to $S^k \times I \times \{1\}$ is the required homotopy. Now, it is clear that $(p_{B,A})_{k*}$ is a zero map. \square

In this way, to every one-ended ANR, M , we can associate an element $\{(M_A, x_A), [p_{B,A}], c_M^*\}$ of $\text{pro-}\mathcal{W}_0$, where \mathcal{W}_0 is a homotopy category of pointed spaces having the homotopy type of a pointed CW complex. Hence, we can get analogues of Hurewicz's isomorphism theorem and Whitehead's theorem since they have been proved (see [20]) for any object in $\text{pro-}\mathcal{W}_0$.

CHAPTER IX

HOMOTOPY EQUIVALENCES AT ∞

In this chapter, we shall define certain classes of proper maps and investigate their properties from two points of view. We shall first look for properties preserved under these maps and then try to decide under what additional assumptions they will be homotopy equivalences at ∞ .

Definition 9.1. A proper map $f:M \rightarrow N$ between locally compact metric spaces is in the class \mathcal{C} provided for any compactum $A \subseteq N$, there is a larger $B \subseteq N$ such that, given a map $\varphi:P \rightarrow N-B$ of a finite polyhedron P into $N-B$, there is $\tilde{\varphi}:P \rightarrow f^{-1}(N-A)$ with $f \circ \tilde{\varphi} \simeq \varphi$ in $N-A$. By restricting P to have a dimension $\leq n$, we will get a class \mathcal{C}_n . \mathcal{C}_∞ will denote the intersection $\bigcap_{i>0} \mathcal{C}_i$.

One easily verifies that every homotopy domination at ∞ is in the class \mathcal{C} and that the inclusion $\mathbb{R} \hookrightarrow \mathbb{R}^2$ is not in \mathcal{C} .

Lemma 9.2. If $f:M \rightarrow N$ is in \mathcal{C} (\mathcal{C}_n) and M is movable (n -movable), then N is also movable (n -movable).

Proof. Let a compactum $A \subset N$ be given. Since M is movable and $f^{-1}(A)$ is a compactum, there is $B' \supset f^{-1}(A)$ satisfying (M) of Chapter IV. Pick $B \supset f(B')$ in N using the fact $f \in \mathcal{C}$. Then B is a required compactum. Indeed, let $C \supset B$ be an arbitrary compactum in N and $\varphi: P \rightarrow N-B$ a map of a finite polyhedron P . There is $\tilde{\varphi}: P \rightarrow f^{-1}(N-B) \subset M-B'$ with $f \circ \tilde{\varphi} = \varphi$ in $N-B$. But $\tilde{\varphi}$ is in $M-f^{-1}(A)$ homotopic to a map $\varphi^*: P \rightarrow M-f^{-1}(C)$. Therefore, φ is in $N-A$ homotopic to $f \circ \varphi^*: P \rightarrow N-C$. \square

Lemma 9.3. If $f: M \rightarrow N$ is in $\mathcal{C}(\mathcal{C}_n)$ and M is trivial (n -trivial) at ∞ , then N is also trivial (n -trivial) at ∞ .

Proof. Let $A \subset N$ be any compactum, and take a compactum $B' \supset f^{-1}(A)$ such that every map of a sphere S^k (of dimension $k \leq n$) into $M-B'$ is null-homotopic in $M-f^{-1}(A)$. Select $B \supset f(B')$ as in the previous lemma. Clearly, every map of a sphere S^k (of dimension $k \leq n$) into $N-B$ is null-homotopic in $N-A$. \square

Lemma 9.4. If $f: M \rightarrow N$ is in \mathcal{C}_n and N is n -tame at ∞ , then f is in \mathcal{C} .

Proof. Let $A \subset N$ be a compactum. Since $f \in \mathcal{C}_n$, there is $B' \supset A$ such that for any map $\varphi: P \rightarrow N-B'$ of an at most n -dimensional finite polyhedron, we can find

$\tilde{\varphi}: P \rightarrow f^{-1}(N-A)$ with $\varphi \simeq f \circ \tilde{\varphi}$ in $N-A$. Let $B \supset B'$ have the property that the inclusion $N-B \hookrightarrow N-B'$ factors through a finite n -complex K , i.e., a diagram

$$\begin{array}{ccc} N-B & \hookrightarrow & N-B' \\ & \searrow \alpha & \nearrow \beta \\ & K & \end{array}$$

homotopy commutes. Take $\tilde{\beta}: K \rightarrow f^{-1}(N-A)$ for which $f \circ \tilde{\beta} \simeq \beta$ in $N-A$. If $\varphi: P \rightarrow N-B$ is a map of a finite polyhedron P into $N-B$, let $\tilde{\varphi} = \tilde{\beta} \circ \alpha \circ \varphi: P \rightarrow f^{-1}(N-A)$. Then $f \circ \tilde{\varphi} \simeq \varphi$ in $N-A$. \square

Definition 9.5. A proper map $f: M \rightarrow N$ is in a class \mathcal{D} provided, for every compactum $A \subset M$, there is a compactum \tilde{A} in N with $f^{-1}(N-\tilde{A}) \subset M-A$ and such that, for any two maps $\varphi, \psi: P \rightarrow f^{-1}(N-\tilde{A})$ of a finite polyhedron P , the relation $f \circ \varphi \simeq f \circ \psi$ in $N-\tilde{A}$ implies $\varphi \simeq \psi$ in $M-A$. If only polyhedra of dimension $\leq n$ are allowed, we will get a class \mathcal{D}_n . $\mathcal{D}_\infty = \bigcap_{i>0} \mathcal{D}_i$.

Lemma 9.6. If $f: M \rightarrow N$ is in \mathcal{D} (\mathcal{D}_n) and N is trivial (n -trivial) at ∞ , then M is also trivial (n -trivial) at ∞ .

Proof. Let $A \subset M$ be a compactum. Pick $\tilde{A} \subset N$ as in the above definition, and, using triviality of N at ∞ , let $\tilde{A}' \supset \tilde{A}$ be such that every map of a sphere S^k (of

dimension $k \leq n$) into $N-\tilde{A}'$ is null-homotopic in $N-\tilde{A}$. Then $B = f^{-1}(\tilde{A}')$ contains A and every map $\varphi: S^k \rightarrow M-B$ is null-homotopic in $M-A$ because φ and a constant map $\psi: S^k \rightarrow f^{-1}(N-\tilde{A}')$ satisfy $f \circ \varphi \simeq f \circ \psi$ in $N-\tilde{A}$ so that $\varphi \simeq \psi$ in $M-A$. \square

Let us observe that if, for a proper map $f: M \rightarrow N$, there is a proper map $g: N \rightarrow M$ with $g \circ f$ homotopic to id_M at ∞ , then f is in the class \mathcal{B} . Also, the inclusion $\mathbb{R} \hookrightarrow \mathbb{R}^2$ is not in \mathcal{B} .

Definition 9.7. The class of proper maps that are both in \mathcal{C} and \mathcal{B} (\mathcal{C}_n and \mathcal{B}_n) will be denoted by \mathcal{G} (\mathcal{G}_n). $\mathcal{G}_\infty = \bigcap_{i>0} \mathcal{G}_i$.

Note that every homotopy equivalence at ∞ is in the class \mathcal{G} . Let $f: X \rightarrow Y$ be a CE map between Z -sets X, Y in \mathcal{Q} . Take an extension $F: Q \rightarrow Q$ of f such that $F^{-1}(Y) = X$. In [19] (see also [29]), it was proved that $F|_{Q-X}: Q-X \rightarrow Q-Y$ is in the class \mathcal{G}_∞ . The converse statement is clearly false as there are maps that are shape equivalences but not CE.

Theorem 9.8. If $f: M \rightarrow N$ is in the class \mathcal{G} (\mathcal{G}_n) and N is movable (n -movable), then M is movable (n -movable).

Proof. Given a compactum $A \subset M$, take $\tilde{A} \subset N$ according to Definition 9.5. Since N is movable, there

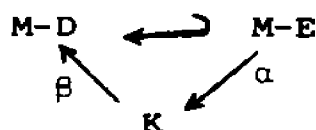
is $\tilde{B} \supset \tilde{A}$ that satisfies (M) of Chapter IV. We claim that $B = f^{-1}(\tilde{B})$ will suffice.

Indeed, $f^{-1}(N-\tilde{A}) \subset M-A$ implies $f^{-1}(\tilde{A}) \supset A$ so $B \supset A$. Let $C \supset B$ be a compactum and $\varphi: P \rightarrow M-B$ a map of a finite polyhedron P into $M-B$. Since $f \in \mathcal{C}$ and $f(C) \supset \tilde{B}$ is a compactum we can find $\tilde{C} \supset f(C)$ such that every map $\kappa: K \rightarrow N-\tilde{C}$ is homotopic in $N-f(C) \subset N-\tilde{A}$ to a map $f \circ \tilde{\kappa}$, where $\tilde{\kappa}: K \rightarrow f^{-1}(N-f(C)) \subset N-C$ and K is a finite polyhedron.

Now, $f \circ \varphi: P \rightarrow N-\tilde{B}$ is homotopic in $N-\tilde{A}$ to a map $\tilde{\psi}: P \rightarrow N-\tilde{C}$. But there is $\psi: P \rightarrow f^{-1}(N-f(C)) \subset N-C$ with $f \circ \psi \simeq \tilde{\psi}$ in $N-f(C) \subset N-\tilde{A}$. Hence, $f \circ \varphi \simeq f \circ \psi$ in $N-A$ so that $\varphi \simeq \psi$ in $M-A$ by the choice of \tilde{A} . \square

Theorem 9.9. Let $f: M \rightarrow N$ be a homotopy domination at ∞ . If f is in the class \mathcal{D}_∞ and M is both movable and tame at ∞ , then f is a homotopy equivalence at ∞ .

Proof. Take a proper map $g: N \rightarrow M$ such that $f \circ g$ is homotopic at ∞ to id_N . We will show that $g \circ f$ is homotopic at ∞ with id_M . Let $A \subset M$ be a compactum. Pick a compactum B_1 in N such that $f^{-1}(N-B_1) \subset M-A$ and $g(N-B_1) \subset M-A$. Since M is movable, there is a compactum $D \supset A_1$ that satisfies (M) of Chapter IV with respect to A_1 . M , being tame at ∞ , gives us a compactum $E \supset D$ so that the diagram



homotopy commutes for some finite complex K . Since $f \in \mathcal{D}_\infty$ there is a compactum B_2 with $f^{-1}(N-B_2) \subset M-E$ and any two maps $\varphi, \psi: K \rightarrow f^{-1}(N-B_2)$ satisfying $f \circ \varphi \simeq f \circ \psi$ in $N-B_2$ are homotopic in $M-E$. Take a compactum $B_3 \supset B_2$ so that $f \circ g|_{N-B_3}$ is homotopic in $N-B_2$ to the inclusion $i: N-B_3 \hookrightarrow N-B_2$. Then $B = f^{-1}(B_3)$ is our required compactum. Indeed, $\beta: K \rightarrow M-D$ is homotopic in $M-A$ to a map $\beta': K \rightarrow M-B$. Observe that $f \circ (g \circ f) \circ \beta' = (f \circ g) \circ (f \circ \beta')$ is homotopic in $N-B_2$ to the map $f \circ \beta'$. Hence, the choice of B_2 gives us that $g \circ f \circ \beta'$ and β' are homotopic in $M-E$. But, then $g \circ f \circ \beta'$ and $g \circ f \circ \beta$ are homotopic in $g \circ f(M-D) \subset M-A$, and therefore $g \circ f \circ \beta$ and β are homotopic in $M-A$. The compositions $g \circ f \circ \beta \circ \alpha$ and $\beta \circ \alpha$ are still homotopic in $M-A$. From here, one easily sees that $g \circ f|_{M-B}$ is homotopic to the inclusion $M-B \hookrightarrow M-A$ inside $M-A$. \square

Theorem 9.10. Let $f: M \rightarrow N$ be in $\mathcal{C}_n \cap \mathcal{B}$. Suppose M is tame at ∞ and N is n -tame at ∞ . Then M is also n -tame at ∞ .

Proof. Take an arbitrary compactum $A \subset M$ and select $A_1 \subset N$ satisfying 9.5, $A_2 \supset A_1$ that satisfies 9.1 with respect to A_1 , and $A_3 \supset A_2$ so that

$$\begin{array}{ccc}
 N-A_3 & \xrightarrow{i} & N-A_2 \\
 \searrow \alpha & & \nearrow \beta \\
 & K^n &
 \end{array}$$

homotopy commutes for some finite n -complex K . Put $\tilde{B} = f^{-1}(A_3) \supset A$. Then our required compactum $B \supset \tilde{B}$ is chosen in such a way that

$$\begin{array}{ccc}
 M-B & \xrightarrow{j} & M-\tilde{B} \\
 \searrow \pi & & \nearrow \rho \\
 & L &
 \end{array}$$

homotopy commutes for some finite complex L . Let $\tilde{\beta}: K \rightarrow f^{-1}(N-A_1)$ have the property $f \circ \tilde{\beta} \simeq \beta$ in $N-A_1$. Consider maps $\tilde{\alpha} = \alpha \circ f \circ j$ and $\tilde{\beta}$. It suffices to prove that $\tilde{\beta} \circ \tilde{\alpha} \simeq j$ in $M-A$. To this end, observe that ρ and $\beta \circ \alpha \circ f \circ \rho$ are two maps of L into $f^{-1}(N-A_1)$ with $f \circ \rho$ and $f \circ \tilde{\beta} \circ \alpha \circ f \circ \rho$ homotopic in $N-A_1$ because $f \circ \tilde{\beta} \simeq \beta$ in $N-A_1$ and $\beta \circ \alpha \simeq i$ in $N-A_2 \subset N-A_1$. The choice of A_1 implies $\rho \simeq \tilde{\beta} \circ \alpha \circ f \circ \rho$ in $M-A$. But, then $\rho \circ \pi \simeq \tilde{\beta} \circ (\alpha \circ f) \circ (\rho \circ \pi)$ in $M-A$. Now, it is easy to see that this gives $\tilde{\beta} \circ (\alpha \circ f \circ j) \simeq j$. \square

Lemma 9.11. Let $f: M \rightarrow N$ be in \mathcal{C} and assume M is n -tame at ∞ and N is tame at ∞ . Then N is n -tame at ∞ .

Proof. Let $A \subset N$ be a compactum. For $A' = f^{-1}(A)$ find a compactum $B' \supset A'$ and a finite n -complex K so that

$$\begin{array}{ccc}
 M-B' & \xhookrightarrow{j} & M-A' \\
 \searrow \rho & & \nearrow \pi \\
 & K &
 \end{array}$$

homotopy commutes. Let B_1 be such that $f^{-1}(N-B_1) \subset M-B'$, $B_2 \supset B_1$ is picked using $f \in \mathcal{C}$, and a required $B \supset B_2$ makes

$$\begin{array}{ccc}
 N-B_2 & \xhookrightarrow{\quad} & N-B \\
 \nwarrow \beta & & \searrow \alpha \\
 & L &
 \end{array}$$

homotopy commutative for some finite complex L . Now, take a map $\tilde{\beta}: L \rightarrow f^{-1}(N-B_2)$ for which $f \circ \tilde{\beta} \simeq \beta$ in $N-B_1$. Consider maps $\alpha' = \rho \circ \tilde{\beta} \circ \alpha: N-B \rightarrow K$ and $\beta' = f \circ \pi: K \rightarrow N-A$. We will show, $\beta' \circ \alpha' \simeq i$, the inclusion $N-B \hookrightarrow N-A$, in $N-A$. Indeed, $\beta' \circ \alpha' = f \circ (\pi \circ \rho) \circ (\tilde{\beta} \circ \alpha)$ is homotopic in $f(M-A') \subset N-A$ to $f \circ j \circ (\tilde{\beta} \circ \alpha)$, where j is the inclusion $M-B' \hookrightarrow M-A'$. But $f \circ j \circ \tilde{\beta} \circ \alpha = f \circ \tilde{\beta} \circ \alpha$ is homotopic in $N-B_1$ to $\beta \circ \alpha$ and, therefore, to i as claimed. \square

Corollary 9.12. Let $f: M \rightarrow N$ be an \mathcal{C} -map between locally compact metric spaces that are tame at ∞ . Then M is n -tame at ∞ if and only if N is n -tame at ∞ . \square

Theorem 9.13. Let $f: M \rightarrow N$ be in \mathcal{C}_∞ and suppose N is both movable and tame at ∞ while M is n -tame at ∞ .

Then N is n -tame at ∞ .

Proof. Let $A \subset N$ be a compactum. For $A' = f^{-1}(A)$ find a compactum $B' \supset A'$ and a finite n -complex K so that

$$\begin{array}{ccc} M-B' & \xrightarrow{j_1} & M-A' \\ \alpha \searrow & & \nearrow \beta \\ & K & \end{array}$$

homotopy commutes. Since N is movable, there is a compactum $C \supset A$ so that condition (M) of Chapter IV holds. Let $D \supset C$ and a finite complex L^k be such that

$$\begin{array}{ccc} N-D & \xrightarrow{i_1} & N-C \\ \pi \searrow & & \nearrow \rho \\ & L & \end{array}$$

commutes up to homotopy and $f^{-1}(N-D) \subset M-B'$. Finally, our required compactum $B \supset D$ is picked using the fact $f \in \mathcal{C}_k$. We will show that the inclusion $N-B \xrightarrow{i} N-A$ factors through the complex K . Take a map $\rho': L \rightarrow N-B$ homotopic in $N-A$ to ρ . The choice of B gives us a map $\tilde{\rho}': L \rightarrow f^{-1}(N-D) \subset M-B'$ with $f \circ \tilde{\rho}' \simeq \rho'$ in $N-D$. Define $a: N-B \rightarrow K$ by $a = \alpha \circ \tilde{\rho}' \circ (\pi|_{N-B})$ and $b: K \rightarrow N-A$ by $b = f \circ \beta$. One can easily check that $b \circ a \simeq i$ in $N-A$. \square

Theorem 9.14. Let $f: M \rightarrow N$ be in $\mathcal{C}_n \cap \mathcal{D}_\infty$ and assume M is movable and tame at ∞ while N is n -tame

at ∞ . Then M is also n -tame at ∞ .

Proof. Let $A \subset M$ be an arbitrary compactum and take $C \supset A$ so that the condition (M) from Chapter IV holds. Next, pick a compactum $D \supset C$ and a finite complex L^k for which the diagram

$$\begin{array}{ccc} M-C & \xleftarrow{i} & M-D \\ & \searrow \rho & \nearrow \pi \\ & L & \end{array}$$

homotopy commutes. Now, we select $A_1 \subset N$ using $f \in \mathcal{B}_k$ (with respect to D) and then $A_2 \supset A_1$ using $f \in \mathcal{C}_n$. Finally, $A_3 \supset A_2$ is such that

$$\begin{array}{ccc} N-A_2 & \xleftarrow{j} & N-A_3 \\ & \searrow \beta & \nearrow \alpha \\ & K & \end{array}$$

commutes for some finite n -complex K , and $B = f^{-1}(A_3)$ is the required compactum. Indeed, let $\tilde{\beta}: K \rightarrow f^{-1}(N-A_1)$ satisfy $f \circ \tilde{\beta} \simeq \beta$ in $N-A_1$. Take $(\tilde{\beta})': K \rightarrow M-B$ and $\rho': L \rightarrow M-B$ where $(\tilde{\beta})' \simeq \tilde{\beta}$ and $\rho' \simeq \rho$ with both homotopies in $M-A$. Define $a: M-B \rightarrow K$ as $a = \alpha \circ (f|_{M-B}) \circ (\rho' \circ \pi|_{M-B})$. We will show that $\tilde{\beta} \circ a$ is homotopic to the inclusion $i: M-B \hookrightarrow M-A$ in $M-A$. Observe that $\tilde{\beta} \circ a \circ f \circ \rho'$ and ρ' are two maps of L into $f^{-1}(N-A_1)$ with

$f \circ (\tilde{\beta} \circ \alpha \circ f \circ \rho')$ $\simeq f \circ \rho'$ in $N-A_1$, by the choice of A_1 because $f \circ \tilde{\beta} \simeq \beta$ in $N-A_1$, $\beta \circ \alpha \simeq j$ in $N-A_2$. The choice of A_1 gives us $\tilde{\beta} \circ \alpha \circ f \circ \rho' \simeq \rho'$ in $M-D$. But then $\tilde{\beta} \circ \alpha \circ f \circ \rho' \circ \pi|_{M-B} \simeq \rho' \circ \pi|_{M-B} \simeq \rho' \circ \pi|_{M-B}$ in $M-D$. Hence, $\tilde{\beta} \circ \alpha \simeq \rho' \circ \pi|_{M-B} \simeq \rho \pi|_{M-B} \simeq i$ in $M-A$, as claimed. \square

Lemma 9.15. Classes \mathcal{C} , \mathcal{C}_n , \mathcal{D} , \mathcal{D}_n are closed under composition of maps.

Proof. Let $f: M \rightarrow N$ and $g: N \rightarrow P$ be proper maps between locally compact metric spaces. We will prove that (a) $f, g \in \mathcal{C}$ implies $g \circ f \in \mathcal{C}$, and (b) $f, g \in \mathcal{D}$ implies $g \circ f \in \mathcal{D}$.

(a) Let $A \subset P$ be a compactum. Since $g^{-1}(A)$ is a compactum in N and $f \in \mathcal{C}$, there is $C \supset g^{-1}(A)$ so that for any map $\tilde{\varphi}: K \rightarrow N-C$ of a finite complex K there is a map $\tilde{\varphi}: K \rightarrow f^{-1}(N-g^{-1}(A)) = M - (g \circ f)^{-1}(A)$ with $f \circ \tilde{\varphi} \simeq \tilde{\varphi}$ in $N-C$. Since $g \in \mathcal{C}$ and $g(C)$ is a compactum in P , there is $B \supset g(C) \supset A$ so that for any map $\varphi: K \rightarrow P-B$ of a finite complex K we can find $\tilde{\varphi}: K \rightarrow g^{-1}(P-g(C)) \subset N-C$ such that $g \circ \tilde{\varphi} \simeq \varphi$ in $P-g(C) \subset P-A$. Hence, $(g \circ f) \circ \tilde{\varphi} \simeq g \circ \tilde{\varphi} \simeq \varphi$ in $P-A$.

(b) Since $f \in \mathcal{D}$, there is a compactum $\tilde{A} \subset N$ satisfying 9.5 with respect an arbitrary compactum A in M . Then use $g \in \mathcal{D}$ and pick $\tilde{A} \subset P$ with respect to \tilde{A} . Observe that $g^{-1}(P-\tilde{A}) \subset N-\tilde{A}$ and $f^{-1}(N-\tilde{A}) \subset M-A$, hence,

$f^{-1}g^{-1}(P-\tilde{A}) \subset M-A$ and given $\varphi, \psi: K \rightarrow f^{-1}g^{-1}(P-\tilde{A})$ with $g \circ f \circ \varphi \simeq g \circ f \circ \psi$ in $P-\tilde{A}$, then $f \circ \varphi \simeq f \circ \psi$ in $N-\tilde{A}$ and, finally, $\varphi \simeq \psi$ in $M-A$. \square

We will say that a locally compact metric space M has type n at ∞ provided for any compactum $A \subset M$ there is a larger compactum B so that every two maps $\varphi, \psi: X \rightarrow M-B$ that are n -homotopic in $M-B$ (i.e., for every map $\alpha: K \rightarrow X$ of an at most n -dimensional finite complex K , $\varphi \circ \alpha \simeq \psi \circ \alpha$ in $M-B$) are homotopic in $M-A$.

Note 9.16. If $f: M \rightarrow N$ is in \mathcal{D}_n and M has type n at ∞ , then f is in \mathcal{D} .

Proof. Let A be a compactum in M and take $B \supset A$ according to above definition. Now, choose a compactum \tilde{B} in N using $f \in \mathcal{D}_n$. Assume $\varphi, \psi: P \rightarrow f^{-1}(N-\tilde{B})$ satisfy $f \circ \varphi \simeq f \circ \psi$ in $N-\tilde{B}$. If $\alpha: K \rightarrow P$ is an arbitrary map of an at most n -dimensional finite complex K into P . Then $f \circ \varphi \circ \alpha \simeq f \circ \psi \circ \alpha$ in $N-\tilde{B}$, so the choice of \tilde{B} gives $\varphi \circ \alpha \simeq \psi \circ \alpha$ in $M-B$. In other words, φ and ψ are n -homotopic in $M-B$. Therefore, $\varphi \simeq \psi$ in $M-A$. \square

Lemma 9.17. Let $f: M \rightarrow N$ be a proper map between locally compact metric spaces. Then the natural projection $\pi: Z_f \rightarrow N$ of a mapping cylinder Z_f of f onto N is in the class \mathcal{D} .

Proof. (a) π is in \mathcal{C} . If A is a compactum in N and $\varphi: P \rightarrow N-A$ is a map of a finite complex into $N-A$, let $\tilde{\varphi}: P \rightarrow Z_f^{-1}\pi^{-1}(A) = \pi^{-1}(N-A)$ be the composition $j \circ \varphi$, where $j: N \hookrightarrow Z_f$ is the natural inclusion. Clearly, $\pi \circ j = \text{id}_N$.

(b) π is in \mathcal{B} . Let $A \subset Z_f$ be a compactum. Put $\tilde{A} = \pi(A) \subset N$. If $\varphi, \psi: P \rightarrow \pi^{-1}(N-\tilde{A})$ are two maps of a finite complex P such that $\pi \circ \varphi \simeq \pi \circ \psi$ in $N-\tilde{A}$, then $\varphi \simeq \psi$ in Z_f-A because $\varphi \simeq j\pi \circ \varphi$ in Z_f-A and $\psi \simeq j\pi \circ \psi$ in Z_f-A . \square

Lemma 9.18. Let $f: M \rightarrow N$ and $g: N \rightarrow P$ be proper maps between locally compact metric spaces. If g is in \mathcal{B} (\mathcal{B}_n) and $h = g \circ f$ is in \mathcal{C} (\mathcal{C}_n), then f is in \mathcal{C} (\mathcal{C}_n).

Proof. Let a compactum $A \subset N$ be given. Since $g \in \mathcal{B}$ we can find a compactum $\tilde{A} \subset P$ such that $g^{-1}(P-\tilde{A}) \subset N-A$ and whenever $\varphi, \psi: K \rightarrow g^{-1}(P-\tilde{A})$ are two maps of a finite complex with $g \circ \varphi \simeq g \circ \psi$ in $P-\tilde{A}$, then $\varphi \simeq \psi$ in $N-A$. Note that $g(A) \subset \tilde{A}$. Since $h \in \mathcal{C}$, there is $B' \supset \tilde{A}$ such that every map $\psi: K \rightarrow P-B'$ can be up to homotopy in $P-\tilde{A}$ represented as $h \circ \tilde{\psi}$, where $\tilde{\psi}: K \rightarrow h^{-1}(P-\tilde{A}) \subset f^{-1}(N-A)$. Let $B = g^{-1}(B') \supset A$. If $\varphi: K \rightarrow N-B$ is an arbitrary map of a finite complex, then $\psi = g \circ \varphi: K \rightarrow P-B'$ is homotopic in $P-\tilde{A}$ to $g \circ f \circ \tilde{\psi}$, where $\tilde{\psi}: K \rightarrow h^{-1}(P-\tilde{A})$. By the choice of \tilde{A} , $f \circ \tilde{\psi} \simeq \varphi$ in $N-A$. \square

Lemma 9.19. Let $f: M \rightarrow N$ and $g: N \rightarrow P$ be proper

maps between locally compact metric spaces. If $h = g \circ f$ is in the class \mathcal{D} (\mathcal{D}_n), then $f \in \mathcal{D}$ ($f \in \mathcal{D}_n$).

Proof. Take a compactum $A \subset M$. Since $h \in \mathcal{D}$, there is $A' \subset P$ such that $h^{-1}(P-A') \subset M-A$ and for any two maps $\varphi, \psi: K \rightarrow h^{-1}(P-A')$ of a finite complex with $h \circ \varphi \simeq h \circ \psi$ in $P-A'$ it follows $\varphi \simeq \psi$ in $M-A$. Let $\tilde{A} = g^{-1}(A')$. One easily verifies that 9.5 holds. \square

Combining Lemma 9.15 with Lemmas 9.17, 9.18 and 9.19 gives.

Theorem 9.20. A proper map $f: M \rightarrow N$ is in a class \mathcal{X} ($\mathcal{X} = \mathcal{C}, \mathcal{C}, \mathcal{C}_n, \mathcal{D}, \mathcal{D}_n, \mathcal{E}, \mathcal{E}_n$) iff the inclusion $i: M \hookrightarrow Z_f$ is in a class \mathcal{X} .

Theorem 9.21. Let $f: M \rightarrow N$ be a proper map between finite-dimensional locally compact ANR's. If $f \in \mathcal{E}$, then f is a homotopy equivalence at ∞ .

Proof. There is no loss of generality if we assume that M, N are locally finite CW complexes [51]. Now, the idea is to verify that conditions $(\pi_1)_\infty$ and $(\pi_*)_\infty$ in [42] hold for the inclusion $X \hookrightarrow Y$, where X is a copy of M and Y is a mapping cylinder of f .

Since $X \hookrightarrow Y$ is of class \mathcal{E} (Corollary 9.20) for any cofinite subcomplex Y' of Y (i.e., $Y-Y'$ has compact closure) there exists a cofinite $Y'' \subset Y$ so that (a) every

map $\varphi: K \rightarrow Y''$ is homotopic in Y' to a map $\tilde{\varphi}: K^n \rightarrow X'$, where K is any finite complex and $X' = X \cap Y'$, and (b) any two maps $\varphi, \psi: K \rightarrow X'' = X \cap Y''$ that are homotopic in Y'' are homotopic in X' .

One can easily check that (a) and (b) imply that $(\pi_1)_\infty$ holds. Some diagram chasing will be necessary to see that the condition

$(\pi_\star)_\infty$: For some cofinite subcomplex Y' in each neighborhood of ∞ in Y , there is a cofinite $Y'' \subset Y'$ so that the map $\pi_\star(Y'', X'') \rightarrow \pi_\star(Y', X')$ is zero, is satisfied.

Let Y' be given inside an arbitrary neighborhood of ∞ . Take cofinite subcomplexes $Y_0 = Y' \supset Y_1 \supset Y_2 \supset Y_3 \supset Y_4 = Y''$ such that (a) and (b) hold for each pair (Y_i, Y_{i+1}) ($i = 0, 1, 2, 3$). Consider homotopy ladders for pairs $(Y_i, X_i = X \cap Y_i)$.

$$\begin{array}{ccccccccc}
 \pi_n(X_4) & \xrightarrow{k_4} & \pi_n(Y_4) & \xrightarrow{\ell_4} & \pi_n(Y_4, X_4) & \xrightarrow{\partial} & \pi_{n-1}(X_4) & \xrightarrow{k_4} & \pi_{n-1}(Y_4) \\
 \downarrow i_{43} & & \downarrow j_{43} & & \downarrow m_{43} & & \downarrow i_{43} & & \downarrow j_{43} \\
 \pi_n(X_3) & \xrightarrow{k_3} & \pi_n(Y_3) & \xrightarrow{\ell_3} & \pi_n(Y_3, X_3) & \xrightarrow{\partial} & \pi_{n-1}(X_3) & \xrightarrow{k_3} & \pi_{n-1}(Y_3) \\
 \downarrow i_{32} & & \downarrow j_{32} & & \downarrow m_{32} & & \downarrow i_{32} & & \downarrow j_{32} \\
 \pi_n(X_2) & \xrightarrow{k_2} & \pi_n(Y_2) & \xrightarrow{\ell_2} & \pi_n(Y_2, X_2) & \xrightarrow{\partial} & \pi_{n-1}(X_2) & \xrightarrow{k_2} & \pi_{n-1}(Y_2) \\
 \downarrow i_{21} & & \downarrow j_{21} & & \downarrow m_{21} & & \downarrow i_{21} & & \downarrow j_{21} \\
 \pi_n(X_1) & \xrightarrow{k_1} & \pi_n(Y_1) & \xrightarrow{\ell_1} & \pi_n(Y_1, X_1) & \xrightarrow{\partial} & \pi_{n-1}(X_1) & \xrightarrow{k_1} & \pi_{n-1}(Y_1) \\
 \downarrow i_{10} & \nearrow \beta & \downarrow j_{10} & & \downarrow m_{10} & & \downarrow i_{10} & \nearrow \beta & \downarrow j_{10} \\
 \pi_n(X_0) & \xrightarrow{k_0} & \pi_n(Y_0) & \xrightarrow{\ell_0} & \pi_n(Y_0, X_0) & \xrightarrow{\partial} & \pi_{n-1}(X_0) & \xrightarrow{k_0} & \pi_{n-1}(Y_0)
 \end{array}$$

Here i 's, j 's and k 's are induced by inclusions. We will show that $m_{40}:\pi_n(Y'',X'') \rightarrow \pi_n(Y',X')$ is a zero map, for every n . To this end, first define a function $\beta:\pi_n(Y_2) \rightarrow \pi_n(X_0)$ such that $k_0 \circ \beta = j_{20}$ as follows: if $[\varphi] \in \pi_n(Y_2)$ has a representative φ , by (a), there is a map ψ into X_1 homotopic in Y_1 to φ ; put $\beta([\varphi]) = i_{10}([\psi])$. Now, take $[\varphi] \in \pi_n(Y_4, X_4)$. Since $k_4 \circ ([\varphi]) = 0 = k_4(0)$, by (b), $i_{43} \circ \partial([\varphi]) = i_{43} \circ \partial(0)$ or $\partial \circ m_{43}([\varphi]) = 0$. Hence, $i_{32} \circ \partial \circ m_{43}([\varphi]) = 0$. By commutativity, $\partial \circ m_{42}([\varphi]) = 0$ which means that $\text{Im}(m_{42}) \subset \ker \partial$. But $\ker \partial = \text{Im } \ell_2$ so that $m_{43}([\varphi]) = \ell_2([\psi])$, for some element $[\psi] \in \pi_n(Y_2)$. Now, $m_{40}([\varphi]) = m_{20} \ell_2([\psi]) = \ell_0 \circ j_{20}([\psi]) = \ell_0 \circ k_0 \circ \beta([\psi]) = 0$ since the last row is exact.

Theorem 9.22. Let $f:M \rightarrow N$ be a proper map for which there is a proper map $g:N \rightarrow M$ with $g \circ f$ homotopic at ∞ to id_M . If $f \in \mathcal{C}$ and N is tame at ∞ , then f is a homotopy equivalence at ∞ .

Proof. We will show that $f \circ g$ is homotopic at ∞ to id_N . Let $A \subset N$ be an arbitrary compactum. $A' = f^{-1}(A)$ is a compactum in M . Pick $B' \supset A'$ so that $g \circ f|_{M-B'}$ is homotopic in $M-A'$ to the inclusion $i:M-B' \rightarrow M-A'$. Let $B_1 = g^{-1}(B')$. $B_2 \supset B_1$ is chosen in such a way that every map $\varphi:P \rightarrow N-B_2$ of a finite complex P is homotopic in $N-B_1$ to $f \circ \tilde{\varphi}$, where $\tilde{\varphi}:P \rightarrow f^{-1}(N-B_1)$. Then our required

compactum B is selected so that the inclusion

$j: N-B \hookrightarrow N-B_2$ factors up to homotopy through a finite complex K

$$\begin{array}{ccc} N-B & \xhookrightarrow{j} & N-B_2 \\ & \searrow \alpha & \nearrow \beta \\ & K & \end{array} .$$

By assumption, there is $\tilde{\beta}: K \rightarrow f^{-1}(N-B_1)$ such that $f \circ \tilde{\beta} \sim \beta$ in $N-B_1$. Then $g \circ f \tilde{\beta} \sim g \circ \beta$ in $M-B'$. But $g \circ f|_{M-B'} \sim i$ in $M-A'$ so that $f \circ g \circ \beta \sim f \tilde{\beta} \sim \beta$ in $N-A$. Therefore, $f \circ g \circ \beta \circ \alpha \sim \beta \circ \alpha$ in $N-A$ or $f \circ g|_{N-B} \sim$ the inclusion $N-B \hookrightarrow N-A$ in $N-A$. \square

Lemma 9.23. Let $f: M \rightarrow N$ be of class \mathcal{C}_∞ with N both movable and tame at ∞ . Then f is in \mathcal{C} .

Proof. Take an arbitrary compactum $A \subset N$ and pick compacta B_1 and B_2 such that B_1 satisfies condition (M) from Chapter IV with respect to A while

$$\begin{array}{ccc} N-B_1 & \xleftarrow{\quad} & N-B_2 \\ & \nwarrow \beta & \swarrow \alpha \\ & K & \end{array}$$

homotopy commutes, for some finite complex K . Since

$f \in \mathcal{C}_{\dim K}$ we can select our required B for which 9.1

holds with respect to B_2 . Assume $\varphi: P \rightarrow N-B$ is a map of

an arbitrary finite complex P . Let $\beta': K \rightarrow N-B$ be homotopic in $N-A$ to β , and $\tilde{\beta}': K \rightarrow f^{-1}(N-B_2) \subset f^{-1}(N-A)$ a map with $f \circ \tilde{\beta}' \simeq \beta'$ in $N-B_2$. Then one easily sees that $\tilde{\varphi} = \tilde{\beta}' \circ (\alpha|_{N-B}) \circ \varphi$ has the property $f \circ \tilde{\varphi} \simeq \varphi$ in $N-A$. \square

Theorem 9.24. Let $f: M \rightarrow N$ be a domination at ∞ and suppose $f \in \mathcal{D}$. If M is tame at ∞ , then f is an equivalence at ∞ .

Proof. Take a proper map $g: N \rightarrow M$ such that $f \circ g$ is homotopic at ∞ to id_N . We will demonstrate that $g \circ f$ is homotopic at ∞ to id_M .

Let $A \subset M$ be any compactum. Pick a compactum \tilde{A} in N using $f \in \mathcal{D}$, and a compactum $\tilde{B} \supset \tilde{A}$ for which $f \circ g|_{N-\tilde{B}}$ is homotopic in $N-\tilde{A}$ to the inclusion $i: N-\tilde{B} \hookrightarrow N-\tilde{A}$. Let $B_1 \supset A$ satisfy $f(M-B_1) \subset N-\tilde{B}$ and select our required compactum $B \supset B_1$ in such a way that the diagram

$$\begin{array}{ccc} M-B & \xrightarrow{j} & M-B_1 \\ & \searrow \alpha & \nearrow \beta \\ & K & \end{array}$$

homotopy commutes for some finite complex K . Since $f \circ (g \circ f \beta) = (f \circ g) \circ (f \circ \beta)$ is homotopic in $N-\tilde{A}$ to $f \circ \beta$, by the choice of \tilde{A} we have, $g \circ f \circ \beta \simeq \beta$ in $M-A$. Hence $g \circ f \circ \beta \circ \alpha \simeq \beta \circ \alpha$ in $M-A$. But $\beta \circ \alpha \simeq j$ in $M-B_1$ and $g \circ f(M-B_1) \subset M-A$ so that $g \circ f \simeq i$ in $M-A$. \square

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Title of Thesis: Infinite-Dimensional Topology and the Theory of Shape

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Date of Examination:

December 1, 1975